14.452 Recitation Notes:
1. Gorman's Aggregation Theorem
2. Normative Representative Household Theorem
3. Representative Firm Theorem
(Recitation 2 on November 6, 2009)
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Gorman's Aggregation Theorem: Setup

- Consider an economy with goods $j \in N$ and individuals $h \in \mathcal{H}$, with Walrasian demand functions $x^h\left(p, w^h\right)$. Consider the aggregate demand $\int_{\mathcal{H}} x^h\left(p, w^h\right) dh$, which is, in general, a function of p and the income distribution $\left(w^h\right)_{\mathcal{H}}$.
- When the aggregate demand can be written as a function of p and the aggregate income $w \equiv \int_h w^h dh$, then we say that there exists a representative household. Put differently, a representative household exists iff there exists a Walrasian demand function $x\left(p, w\right)$ such that

$$x\left(p, \int_h w^h dh\right) = \int_{\mathcal{H}} x^h\left(p, w^h\right) dh.$$

Gorman's Aggregation Theorem

Theorem

(Gorman's Aggregation Theorem) *Consider an economy with $N < \infty$ of commodities and a set $\mathcal{H}$ of households. Suppose that the preferences of each household $h \in \mathcal{H}$ can be represented by an indirect utility function of the form*

$$v^h\left(p, y^h\right) = a^h\left(p\right) + b\left(p\right) w^h, \tag{1}$$

and that each household $h \in \mathcal{H}$ has a positive demand for each commodity, then these preferences can be aggregated and represented by those of a representative household, with indirect utility

$$v\left(p, y\right) = a\left(p\right) + b\left(p\right) w,$$

where $a\left(p\right) \equiv \int_{h \in \mathcal{H}} a^h\left(p\right) dh$, and $w \equiv \int_{h \in \mathcal{H}} w^h dh$ is aggregate income.

Gorman Aggregation Theorem: Proof

- Recall that, from Roy's identity, for each good j,

$$x_j^h\left(p, w^h\right) = -\frac{\frac{dv^h\left(p, w^h\right)}{dp_j}}{\frac{dv^h(p, y)}{dw^h}} = -\frac{da^h\left(p\right)}{dp_j}\frac{1}{b\left(p\right)} - \frac{db\left(p\right)}{dp_j}\frac{1}{b\left(p\right)}w^h.$$

  Add up the demand functions to get

$$\begin{aligned}\int_h x_j^h\left(p, w^h\right) &= -\int_{\mathcal{H}}\frac{da^h\left(p\right)}{dp_j}dh\frac{1}{b\left(p\right)} - \frac{db\left(p\right)}{dp_j}\frac{1}{b\left(p\right)}\int_{\mathcal{H}} w^h dh \\ &\equiv x\left(p, w\right),\end{aligned}$$

  where the second line follows since the first line is a function of $w = \int_{\mathcal{H}} w^h dh$. Moreover, using Roy's identity for $v\left(p, y\right)$ in the problem statement, it can be seen that $x\left(p, w\right)$ is the Walrasian demand corresponding to the indirect utility function $v\left(p, y\right)$, completing the proof.
- Intuition: For each price level p and aggregate wealth distribution $\left(w^h\right)_{\mathcal{H}}$, Engel curves (wealth expansion paths) have the same slope for each individual. Additional income is spent the same way, no matter whom it is allocated to.

Gorman Preferences: Quasilinear

- Suppose preferences are given by

$$U^h(x) = x_1 + \bar{U}^h(x_{2,..,}x_N).$$

  Note that we can always normalize the coefficient in front of x_1 to be equal to 1.

- Allow for negative consumption of x_1, or suppose endowment w^h of each household is sufficiently large that there is an interior solution.
- The consumption of $x_2, ..., x_N$ is independent of w^h, i.e. can be written only as a function of price, $x_2^h(p), ..., x_N^h(p)$. This further implies that $x_1 = \frac{w^h - \sum_{j=2}^N p_j x_j^h(p)}{p_1}$. Using these observations, the indirect utility function is

$$\begin{aligned} v^h(p, w^h) &= x_1(p, w^h) + \bar{U}^h\left(x_2^h(p), .., x_N^h(p)\right) \\ &= \frac{w^h}{p_1} - \sum_{j=2}^N \frac{p_j}{p_1} x_j^h(p) + \bar{U}^h\left(x_2^h(p), .., x_N^h(p)\right), \end{aligned}$$

  which satisfies the Gorman form with $b(p) = \frac{1}{p_1}$ and some $a^h(p)$. Intuition?

Gorman Preferences: CES

- Consider preferences

$$U^h\left(x_1^h, ..., x_N^h\right) = \left[\sum_{j=1}^{N}\left(x_j^h - \xi_j^h\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \qquad (2)$$

where $\sigma \in (0, \infty)$ and ξ_j^h captures whether good j is a necessity.

- Define level of consumption w.r.t. reference point $\hat{x}_j^h = x_j^h - \xi_j^h$. Elasticity of substitution between $\hat{x}_j^h$ and $\hat{x}_{j'}^h$ (with $j \neq j'$) is σ.
- Suppose $\sum_{j=1}^{N} p_j \bar{\xi} < w^h$ so that there is always an interior allocation.

Gorman Preferences: CES

- The indirect utility function is

$$v^h\left(p, w^h\right) = \frac{\left[-\sum_{j=1}^N p_j \xi_j^h + w^h\right]}{\left[\sum_{j=1}^N p_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}. \quad (3)$$

- Satisfies the Gorman form with $b(p) = 1/\left[\sum_{j=1}^N p_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$.
- Intuition: Elasticity of substitution is constant (independent of the identity of household h or the income level w^h). Thus, additional income is spent in the same proportions, regardless whom it is allocated to.
- The utility function that leads to the function in (3) is

$$U(x_1, ..., x_N) = \left[\sum_{j=1}^N \left(x_j - \xi_j\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \quad (4)$$

where $\xi_j = \int_{\mathcal{H}} \xi_j^h dh$.

Normative Representative Household Theorem: Setup

- Gorman preferences are useful since they ensure the existence of a positive representative household. Can we use the aggregated preferences for Normative purposes?
- Note that representative consumer cannot say much about Pareto optimality regarding distribution of a given level of endowment (or production).
- Fortunately, with Gorman preferences, distributional issues are straightforward.
- But rep. consumer has implications of whether aggregate *production decisions* are optimal.
- It turns out that, with Gorman preferences, production decisions obtained from the representative consumer are also optimal for the decentralized economy.

Normative Representative Household Theorem

Theorem

(Existence of a Normative Representative Household) *Consider an economy with a finite number $N < \infty$ of commodities, a set $\mathcal{H}$ of households and a convex aggregate production possibilities set Y. Suppose that the preferences of each household $h \in \mathcal{H}$ can be represented by Gorman form, $v^h(p, w^h) = a^h(p) + b(p) w^h$, where $p = (p_1, ..., p_N)$ is the price vector, and that each household $h \in \mathcal{H}$ has a positive demand for each commodity.*

1. *Then, any feasible allocation that maximizes the utility of the representative household, $v(p, w) = \sum_{h\in\mathcal{H}} a^h(p) + b(p) w$, with $w \equiv \sum_{h\in\mathcal{H}} w^h$, is Pareto optimal.*
2. *Moreover, if there exists a function $\bar{a}(p)$ such that $a^h(p) = a^h + \bar{a}(p)$ for all p and all $h \in \mathcal{H}$, then any Pareto optimal allocation maximizes the utility of the representative household.*

Normative Representative Household Theorem: Intuition

- Part 1 says, a Pareto optimal production decision for the rep. household economy is also Pareto optimal for the decentralized economy (with an appropriate distribution of the output).
- Proof in the textbook and lecture notes.
- Idea of proof: Consider the Pareto problem for the decentralized economy and the representative household economy, written using indirect utility functions (and Roy's identity) to draw upon the Gorman aggregation theorem.
- Writing the problems this way immediately implies that the socially optimal allocation for the rep. household economy is also Pareto optimal for the decentralized economy with Pareto weights $\alpha^h = 1$ for each $h \in \mathcal{H}$.
- This is because, with Gorman preferences, the indirect utility of the representative household is the sum of indirect utilities of the households.
- Converse statement (part 2) is that, the aggregate production corresponding to *any* Pareto optimal allocation in the decentralized economy can be obtained as a Pareto optimal production decision in the rep. household economy. That is, if we solve the latter problem, we are not missing any Pareto optimal production decisions.

Normative Representative Household Theorem: Intuition

- Converse statement is true if $a^h(p)$ can be represented as $a^h + \bar{a}(p)$ (Note that this condition is slightly more general than the version in the book. You can show that the proof generalizes to this case).
- The constant a^h term is not that important because indirect utility is a cardinal utility notion and we can always add/subtract constants. But the second term $\bar{a}(p)$ implies that the intercept of the Engel curves should be the same for all households (for each price level p).
- Intuition is best seen by considering what happens when the intercepts are not the same for two households $h, \tilde{h}$. Suppose h has a greater intercept for cars, $\tilde{h}$ a greater intercept for shoes. If social planner cares more about h (resp. $\tilde{h}$), it will produce more cars (resp. shoes). With rep. households, the social planner will produce roughly similar levels of both goods. While this is Pareto optimal, it is missing the other Pareto optima in which social planner cares more about household h or $\tilde{h}$.

  When intercepts are the same, this is not an issue and the social planners' production decision for rep. household will include all decentralized Pareto optima.

Representative Firm Theorem: Setup

- Consider a production economy with N commodities and represent a firm $f \in \mathcal{F}$ with a production set $Y^f \subset \mathbb{R}^N$.
- Define $Y = \left\{ \sum_{f \in \mathcal{F}} y^f, \ y^f \in Y^f \text{ for each } f \in \mathcal{F} \right\}$ as the aggregate production possibilities.
- The question is: Can we think of this economy as having a single firm choosing a production vector in Y, instead of the many firms $f \in \mathcal{F}$.
- The answer is yes, without stringent assumptions (unlike the representative consumer case).

Representative Firm Theorem

Theorem

(Representative Firm Theorem) *Consider a competitive production economy with* $N \in \mathbb{N} \cup \{+\infty\}$ *commodities and a countable set* $\mathcal{F}$ *of firms, each with a production possibilities set* $Y^f \subset \mathbb{R}^N$. *Let* $p \in \mathbb{R}^N_+$ *be the price vector in this economy and denote the set of profit maximizing net supplies of firm* $f \in \mathcal{F}$ *by* $\hat{Y}^f(p) \subset Y^f$ *(so that for any* $\hat{y}^f \in \hat{Y}^f(p)$, *we have* $p \cdot \hat{y}^f \geq p \cdot y^f$ *for all* $y^f \in Y^f$*). Then, there exists a representative firm with production possibilities set* $Y \subset \mathbb{R}^N$ *and a set of profit maximizing net supplies* $\hat{Y}(p)$ *such that for any* $p \in \mathbb{R}^N_+$, $\hat{y} \in \hat{Y}(p)$ *if and only if* $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$ *for some* $\hat{y}^f \in \hat{Y}^f(p)$ *for each* $f \in \mathcal{F}$.

Representative Firm Theorem: Proof

- **If part:** consider decentralized optima $\hat{y}^f \in \hat{Y}^f(p)$ and show that $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$ is a centralized optimum, i.e. $\hat{y} \in \hat{Y}(p)$.
- Proof by contradiction: Suppose there exists $y \in Y$ such that

$$p \cdot y > p \cdot \hat{y}.$$

Can decompose $y = \sum_{f \in \mathcal{F}} y^f$ for some $y^f \in Y^f$. Using also $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$, the previous displayed equation implies:

$$\sum_{f \in \mathcal{F}} p y^f > \sum_{f \in \mathcal{F}} p \hat{y}^f.$$

- This further implies there exists at least one $\tilde{f} \in \mathcal{F}$ such that $p \cdot y^{\tilde{f}'} > p \cdot \hat{y}^{\tilde{f}'}$, which yields a contradiction to the fact that $\hat{y}^{\tilde{f}}$ is optimal for firm $\tilde{f}$.

Representative Firm Theorem: Proof

- **Only if part:** Consider $\hat{y} \in \hat{Y}(p)$ and show that there exists decentralized optima $\hat{y}^f \in \hat{Y}^f(p)$ such that $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$.
- Since $\hat{y} \in Y$, we can decompose $\hat{y} = \sum_{f \in \mathcal{F}} y^f$ for some $\hat{y}^f \in Y^f$ (not necessarily optimal for firm f).
- Consider decentralized optima $\bar{y}^f \in \hat{Y}^f(p)$. Then,

$$p \cdot \hat{y}^f \leq p \cdot \bar{y}^f \text{ for each } f \in \mathcal{F}, \tag{5}$$

  which further implies

$$p \cdot \hat{y} \leq p \cdot \sum_{f \in \mathcal{F}} \bar{y}^f.$$

- Since $\hat{y}$ is optimal in the aggregate, it must do weakly better than the vector $\bar{y} \equiv \sum_{f \in \mathcal{F}} \bar{y}^f$. So the previous inequality must hold as equality. Then, the inequality in (5) must hold as equality for each $f \in \mathcal{F}$.
- This implies that $\hat{y}^f \in \hat{Y}^f(p)$. The decomposed vectors $\hat{y}^f$ must indeed be optimal for each firm f. This completes the proof.

Representative Firm Theorem: Intuition

- Why does a representative firm exist under much less stringent assumptions?
- Intuitively because there are no wealth effects in the production theory. Firms *do not* start with endowments as households do.
- So the question of how endowments are distributed does not arise. All our troubles with the representative household can be linked to this question of income distribution.

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