14.454 - Macroeconomic Crisis
Problem set 1

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1 Question 1 - Kocherlakota (2000)

Take an economy with a representative, infinitely-lived consumer. The consumer owns a technology with which she produces output (Y) using capital (K) and land (L) according to a production function:

$$Y = F(K, L)$$

where $F(.)$ is increasing, concave and differentiable. Capital fully depreciates after its use $(\delta = 1)$

The consumer is endowed with K_0 units of capital and $L_0 = 1$ units of land at $t = 0$, and has access to an internal land market; i.e. she can buy and sell land in the local market at a price Q_t. The consumer has also access to an international financial market: the consumer can borrow B_t units of consumption goods from international markets at time t, at an interest rate $r > 0$. However, they are constrained on how much they can borrow:

$$B_t \leq B^* \text{ for all } t \tag{1}$$

where B^* is an exogenously given borrowing constraint. The consumer is also born with a debt of $B_0 < B^*$

The consumer has preferences over consumption streams according to the utility function

$$U = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \ln(C_t) \tag{2}$$

where we assume that $\rho = r$. The consumer chooses sequences of consumption $\{C_t\}_{t=0}^{\infty}$, capital $\{K_{t+1}\}_{t=0}^{\infty}$ and land purchases $\{L_t\}_{t=0}^{\infty}$ to maximize (2) given prices $\{Q_t\}_{t=0}^{\infty}$ subject to the budget constraint:

$$C_t + K_{t+1} + Q_t L_{t+1} + B_t (1+r) \leq F(K_t, L_t) + B_{t+1} + Q_t L_t \tag{3}$$

and the borrowing constraint (1). So, the consumer must finance consumption C_t, investment K_{t+1}, interest payments on debt $B_t(1+r)$ and land purchases $Q_t L_{t+1}$ using output, borrowing more funds (B_{t+1}) and selling the land they own at the price price Q_t. To close the economy, assume that the total supply of land remains constant over time and equal to one; i.e. $L_t = 1$ for all $t = 1, 2, \ldots$

(a) Define a competitive equilibrium for this economy

Answer:

A competitive equilibrium for this economy are levels $\{C_t.B_t, K_{t+1}, L_{t+1}\}_{t=0}^{\infty}$ and a land price sequence $\{Q_{t+1}\}_{t=0}^{\infty}$ such that:

1. $\{C_t.B_t, K_{t+1}, L_{t+1}\}_{t=0}^{\infty}$ maximizes (2) given prices $\{Q_{t+1}\}_{t=0}^{\infty}$ and the international interest rate r, subject to constraints $(3), (1)$ and K_0 and B_0 are given.
2. **Equilibrium in internal land market:** $L_{t+1} = 1$ for all t

Note: This is an equilibrium in an open economy, so the usual equilibrium restriction for the bonds market (which is typically $B_t = 0$ for all t) is replaced by the constant interest rate assumption. The idea is that in the global economy the net demand of assets must be zero, but the country we are analyzing is small enough so that it does not affect the equilibrium world interest rate r.

(b) A steady state equilibrium is an equilibrium in which $C_t = C_{ss}, B_t = B_{ss}, Y_t = Y_{ss}, Q_t = Q_{ss}$ and $K_t = K_{ss}$. Using the definition of a competitive equilibrium and the FOC's of the consumer's optimization problem, characterize such an equilibrium, given a borrowing level $B_{ss} < B^*$. Do K_{ss}, Y_{ss} and Q_{ss} depend on the level of B_{ss}?

Answer

The Lagrangian of the consumer problem is

$$\begin{aligned} \mathcal{L} = & \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \ln(C_t) + \\ & \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \lambda_t \left[F(K_t, L_t) + B_{t+1} + Q_t L_t - C_t - K_{t+1} - Q_t L_{t+1} - B_t (1+r)\right] \end{aligned}$$

$$+\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{t}\mu_{t}\left(B^{*}-B_{t+1}\right)$$

so the FOCs of the problem are

$$(C_t):\left(\frac{1}{1+\rho}\right)^{t}\left[\frac{1}{C_t}-\lambda_t\right]=0\Longleftrightarrow\frac{1}{C_t}=\lambda_t \tag{4}$$

$$(K_{t+1}):-\left(\frac{1}{1+\rho}\right)^{t}\lambda_t+\left(\frac{1}{1+\rho}\right)^{t+1}\lambda_{t+1}F_K\left(K_{t+1},L_{t+1}\right)=0\Longleftrightarrow$$

$$\left(\frac{1}{1+\rho}\right)\lambda_{t+1}F_K\left(K_{t+1},L_{t+1}\right)=\lambda_t \tag{5}$$

$$(L_{t+1}):-\left(\frac{1}{1+\rho}\right)^{t}\lambda_t Q_t+\left(\frac{1}{1+\rho}\right)^{t+1}\lambda_{t+1}\left[F_L\left(K_{t+1},L_{t+1}\right)+Q_{t+1}\right]=0\Longleftrightarrow$$

$$\left(\frac{1}{1+\rho}\right)\lambda_{t+1}\left[F_L\left(K_{t+1},L_{t+1}\right)+Q_t\right]=\lambda_t Q_t \tag{6}$$

$$(B_{t+1}):\left(\frac{1}{1+\rho}\right)^{t}\left(\lambda_t+\mu_t\right)-\left(\frac{1}{1+\rho}\right)^{t+1}\left(1+r\right)\lambda_{t+1}=0\Longleftrightarrow$$

$$\lambda_t+\mu_t=\left(\frac{1}{1+\rho}\right)\left(1+r\right)\lambda_{t+1} \tag{7}$$

Substituting $\rho=r$ and that in equilibrium we have that $L_{t+1}=1$ for all t on conditions $(4)-(7)$, we get the following set of dynamic equations:

$$\frac{1}{C_t}=\lambda_t \tag{8}$$

$$\lambda_{t+1}F_K\left(K_{t+1},1\right)=\left(1+r\right)\lambda_t \tag{9}$$

$$\lambda_{t+1}\left[F_L\left(K_{t+1},1\right)+Q_{t+1}\right]=\lambda_t Q_t\left(1+r\right) \tag{10}$$

$$\lambda_t+\mu_t=\lambda_{t+1} \tag{11}$$

In a steady state equilibrium, $C_t=C_{t+1}=C_{ss}$,for all t. From (8) this implies that $\lambda_t=\lambda_{t+1}$ for all t, which implies in (11) that $\mu_t=0$. This means that **a steady state equilibrium is consistent only with a situation in which the borrowing constraint is not binding**. Using the fact that $\lambda_{t+1} = \lambda_t$ in conditions (9) and (10) we get

$$F_K(K_{t+1}, 1) = (1 + r) \text{ for all } t \Longleftrightarrow F_K(K_{ss}, 1) = 1 + r \tag{12}$$

$$F_L(K_{ss}, 1) + Q_{t+1} = Q_t(1 + r) \Longleftrightarrow Q_{ss} = \frac{1}{r} F_L(K_{ss}, 1) \tag{13}$$

Where (12) is simply that the marginal productivity of capital is equal to the return in bonds (which must be true in the optimum) and (13) states that the price of land is equal to the present value of marginal utility of land.

To get the steady state values of output and consumption, we use the budget constraint together with the fact that $B_0 = B_{ss} < B^*$ (if not, then $\mu_t > 0$!) and that $Y_{ss} = F(K_{ss}, 1)$:

$$C_{ss} = F(K_{ss}, 1) - K_{ss} - rB_{ss} \tag{14}$$

Note that the steady state levels of capital, price of land and output are unaffected by the initial borrowing level $B_0 = B_{ss}$ as long as $B_{ss} < B^*$

(c) Assume that $K_0 = K_{ss}$ and $B_0 = B_{ss}$ (so the economy is at its steady state from period $t = 0$). Suppose that the farmer has an unexpected negative shock on debt; i.e. $B_0' = B_0 - \Delta$ and $\Delta = 0$ for all $t \geq 1$. Does this shock have any effect on consumption, output and land prices? What if $B_0' = B_0 + \Delta$ with $\Delta \in (0, B^* - B_0)$? Would it change your results if there was no borrowing constraints?

Answer:

Note that as long as $B_0' < B^*$, then starting in a steady state, we remain there. To see this, see that if the borrowing constraint (1) is not binding at time $t = 0$, then $\mu_t = 0$ and therefore $\lambda_t = \lambda_{t+1}$ which implies that $C_t = C_{t+1} = C_{ss}$, so we have the previous case (that is, we are in the steady state). Therefore, capital, output and the price of land remain constant under this shock. The only difference now is that the consumption level permanently increases if $B_0' = B_0 - \Delta$, by the amount

$$C_{ss}' - C_{ss}' = r\Delta$$

If $B_0' = B_0 + \Delta < B^*$ the same as before holds, so now consumption permanently decreases by $r\Delta$

(d) Suppose that the shock is positive, and that $\Delta > B^* - B_0$ (so that the initial borrowing level exceeds the borrowing limit B^*). What happens with equilibrium output Y_t? Compare it with the situation in which there are no borrowing constraints. Explain.

Answer:

Now, because the constraint (1) is binding, we won't have that $\mu_t = 0$ (if we had, then we would be in a steady state where $B_0 > B^*$, violating the constraint) so we must have $\mu_0 > 0$. At $t = 0$, since capital K_0 is given, then (C_0, K_1) must satisfy the resource constraint

$$F(K_{ss}, 1) - rB^* = C_0 + K_1 < C_{ss} + K_{ss}$$

so initially, at $t = 0$ output remains unchanged, but aggregate expenditures on consumption and investment drop. This will give a reduction in both time $t = 0$ consumption and investment K_1

For $t > 1$ we get that as long as the borrowing constraint is binding ($\mu_t > 0$ and $B_t = B^*$) then

$$C_{t+1} = \left(\frac{\lambda_t + \mu_t}{\lambda_t}\right) C_t > C_t \text{ for all } t$$

From condition (9) at $t > 1$ we get that

$$F_K(K_{t+1}, 1) = (1 + r)\frac{\lambda_t}{\lambda_{t+1}} \Longrightarrow$$

$$F_K(K_{t+1}, 1) = (1 + r)\frac{C_{t+1}}{C_t} > 1 + r = F_K(K_{ss}, 1)$$

and because of marginal returns on capital (i.e. $F_{KK} < 0$), we get that $K_{t+1} < K_{ss}$ for all t. We can show that

$$C_t \nearrow C_{ss} \text{ and } K_t \nearrow K_{ss} \text{ as } t \longrightarrow \infty$$

using arguments about stability of steady state equilibria (with concave production function) of the neoclassical model. Prices of land are given by

$$\lambda_{t+1}[F_L(K_{t+1}, 1) + Q_{t+1}] = \lambda_t Q_t (1 + r) \Longleftrightarrow$$

$$Q_t = \frac{1}{1+r}\frac{\lambda_{t+1}}{\lambda_t}[F_L(K_{t+1}, 1) + Q_{t+1}]$$

and using forward induction (with appropriate boundary conditions) we get that

$$Q_t = \sum_{s=1}^{\infty}\left(\frac{1}{1+r}\right)^s \underbrace{\frac{C_{t+s-1}}{C_{t+s}}}_{u'(C_{t+s})/u'(C_{t+s-1})} F_L(K_{t+s}, 1)$$

which is the present value of the marginal utility of the extra unit of output produced by the last unit of land! Because of homogeneity of degree 1

$$F_L(K_{t+s},1) = F(K_{t+s},1) - K_{t+s}F_K(K_{t+s},1) = F(K_{t+s},1) - (1+r)K_{t+s}\frac{C_{t+s}}{}$$

(e) Suppose now that instead of the borrowing constraint (1), consumers must now collateralize their loans with the value of their land holdings. Namely

$$B_{t+1} \leq Q_t L_{t+1}$$

Moreover, assume that $F(K,L) = K^{\alpha_1}L^{\alpha_2}$ where $\alpha_1, \alpha_2 \geq 0$. Repeat (a) and (b) on this new setting.

Answer:

The Lagrangian of the consumer problem is now

$$\begin{aligned}\mathcal{L} &= \sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t \ln(C_t) + \\ &\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t \lambda_t\left[F(K_t,L_t) + B_{t+1} + Q_tL_t - C_t - K_{t+1} - Q_tL_{t+1} - B_t(1+r)\right]\end{aligned}$$

$$+\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t \mu_t\left(Q_tL_{t+1} - B_{t+1}\right)$$

All conditions remain unchanged, except for the FOC of land, which is now:

$$(L_{t+1}): \left(\frac{1}{1+\rho}\right)^t(\mu_t - \lambda_t)Q_t + \left(\frac{1}{1+\rho}\right)^{t+1}\lambda_{t+1}\left[F_L(K_{t+1},L_{t+1}) + Q_{t+1}\right] = 0 \Longleftrightarrow$$
$$\left(\frac{1}{1+r}\right)\lambda_{t+1}\left[F_L(K_{t+1},L_{t+1}) + Q_{t+1}\right] = (\lambda_t - \mu_t)Q_t$$

If $B_0 < Q_{ss}$ the economy starts in a steady state. Steady state conditions for capital and consumption remain unchanged and the price of land is now given by

$$F_L(K_{ss},1) + Q_{ss} = (1+r)Q_{ss} \Longleftrightarrow Q_{ss} = \frac{F_L(Q_{ss},1)}{r}$$

which is identical as before. So, as long as we start in a steady state that does not bind the collateral constraint, everything is the same (which is not surprising, since both problems are identical if the borrowing constraints are not binding!)

(f) Suppose $K_0 = K_{ss}$ and $B_0 = Q_{ss} + \Delta$, with $\Delta > 0$. Does your conclusions from (d) change?

Answer

Is easy to see how all qualitative results are unchanged. Also, from the FOC on land, we get now that

$$\left(\frac{1}{1+r}\right)\left[F_L\left(K_{t+1}, L_{t+1}\right) + Q_{t+1}\right] = \frac{\lambda_t - \mu_t}{\lambda_t + \mu_t} Q_t \Longrightarrow$$

$$Q_t = \sum_{s=1}^{\infty}\left(\frac{1}{1+r}\right)^s \left[\frac{\lambda_{t+1}}{\lambda_t - \mu_t}\right] F_L\left(K_{t+s}, 1\right)$$

so, Q reacts more rapidly than in the previous model (holding everything else constant)

Note: In Kocherlakota (2000)[1], the author also explores how these two different models create different amplification mechanisms of the credit constraint channel. Which one do you expect to be the one that generates more amplification? Compare to Kiyotaki and Moore (1997)

2 Question 2 (Lorenzoni 2010)

This problem analyzes the welfare effects of a "capital injection"in a model with financial frictions. There are two periods, 0 and 1. Consumers and entrepreneurs have a linear utility function, $c_0 + c_1$. Consumers have a large endowment of consumption goods in each period and a unit endowment of labor in period 1, which they sell inelastically on a competitive labor market at the wage w_1.

Entrepreneurs have a given endowment of consumption goods E_0 and no capital. Then they borrow b_1 and invest k_1. In period 1 they hire workers at the wage w_1 and produce consumption goods according to the Cobb-Douglas production function:

$$y_1 = k_1^{\alpha} l_1^{1-\alpha} \tag{15}$$

The entrepreneurs face the collateral constraint

$$b1 \leq \lambda\left(y1 - w_1 l_1\right)$$

where $\lambda \in (0, 1)$ is a given scalar (think that in period 1 the entrepreneurs can run away with a fraction $(1 - \lambda)$ of the firm's profits). Assume the consumers endowment is large enough that the gross interest rate is always 1 in equilibrium.

[1] Kocherlakota, Narayana, "Creating Business Cycles Through Credit Constraints", *Federal Reserve Bank of Minneapolis Quarterly Review,* Vol. 24, No. 3, Summer 2000, pp. 2–10

The entrepreneur's problem is then

$$\max_{(c_0,c_1)\geq 0, k_1, l_1, b_1} c_0 + c_1$$

$$s.t \quad : \quad \begin{cases} c_0 + k_1 \leq E_0 + b_1 & : (t = 0 \text{ budget constraint}) \\ c_1 + b_1 \leq y_1 - w_1 l_1 & : (t = 1 \text{ budget constraint}) \\ b_1 \leq \lambda (y_1 - w_1 l_1) & : (\text{collateral constraint}) \end{cases}$$

(a) Argue that the entrepreneurs will always choose l_1 to maximize profits in period 1 and that then profits are a linear function of the capital stock k_1, that is:

$$y_1 - w_1 l_1 = R(w_1) k_1$$

where $R(w_1)$ is some (potentially non-linear) function of w_1. Restate the entrepreneur's problem as a simpler linear problem.

Answer:

Is easy to see that the only way profits affect the consumer problem is by making both the $t = 1$ budget constraint and the collateral constraint slacker when profits go up. Therefore, the entrepreneur would want to make profits as big as possible. Therefore, in any optimum the entrepreneur must maximize profits (relate this result to the result shown in 14.121 about the consumer that owns a technology: we showed there that the consumer would always maximize profits).

The profit maximization problem (given a capital stock k_1) is

$$\pi^* = \max_{l_1} y_1 - w_1 l_1 = k_1^{\alpha} l_1^{1-\alpha} - w_1 l_1$$

FOCs:

$$(l_1) : (1-\alpha) k_1^{\alpha} l_1^{*-\alpha} = w_1 \iff l_1^* = \left(\frac{1-\alpha}{w_1}\right)^{\alpha} k_1$$

and plugging in into the profit maximization problem

$$\pi^* = k_1^{\alpha} l_1^{*1-\alpha} - w_1 l_1^* = k_1^{\alpha} \left(\frac{1-\alpha}{w_1}\right)^{\frac{\alpha}{1-\alpha}} k_1^{1-\alpha} - w_1 \left(\frac{1-\alpha}{w_1}\right)^{\alpha} k_1 =$$

$$\underbrace{\left(\frac{1-\alpha}{w_1}\right)^{\alpha} \left[\left(\frac{1-\alpha}{w_1}\right)^{\frac{1}{1-\alpha}} - 1\right]}_{\equiv R(w_1)} k_1 = R(w_1) k_1$$

as we wanted to show. The maximization problem is now

$$\max_{(c_0,c_1)\geq 0,k_1,b_1} c_0 + c_1$$

$$s.t \quad : \begin{cases} c_0 + k_1 \leq E_0 + b_1 & : (t = 0 \text{ budget constraint}) \\ c_1 + b_1 \leq R(w_1)\, k_1 & : (t = 1 \text{ budget constraint}) \\ b_1 \leq \lambda k_1 & : (\text{collateral constraint}) \end{cases}$$

which is a linear program in c_0, c_1, k_1 and l_1

(b) Argue that if

$$\lambda R(w_1) < 1 \leq R(w_1)$$

then the entrepreneur's problem is well defined and the entrepreneur's demand for capital is finite. Derive it.

Answer

Let's study the different cases:

Case 1 $R(w_1) = 1$

Then

$$\max_{(c_0,c_1)\geq 0,k_1,l_1,b_1} c_0 + c_1$$

$$s.t \quad : \begin{cases} c_0 + k_1 \leq E_0 + b_1 & : (t = 0 \textit{ budget constraint}) \\ c_1 + b_1 \leq k_1 & : (t = 1 \textit{ budget constraint}) \\ b_1 \leq \lambda k_1 & : (\textit{collateral constraint}) \end{cases}$$

so the entrepreneur is indifferent between investing in b or k, as long as $b_1 \leq \lambda k_1$

Case 2 $R(w_1) < 1$

In this case, investing is capital gives negative return (because $R(w_1)\, k_1 < k_1$), and hence this technology is dominated by saving in bonds only. Therefore, in the optimum we must have $k_1^* = 0$ and $c_0 + c_1 = E_0$.

Case 3 $\lambda R(w_1) \geq 1$

In this case, we also have that $R(w_1) > 1$. We will show that the problem does not have a solution (i.e. the demand for capital is infinity). For that, fix a capital level $A > 0$ and we will find a feasible plan (c_0, c_1, k_1, b_1) such that capital $k_1 = A$ and utility is

$$U = E_0 + [R(w_1) - 1]\, A$$

As we take $A \longrightarrow \infty$ we see that the problem does not have solution. A feasible plan for implementing $k_1 = A$ (although it is not the only one) is

$$\begin{aligned} k_1 &= b_1 = A \\ c_0 &= E_0 \\ c_1 &= [R(w_1) - 1] A \end{aligned}$$

Is easy to check that it satisfies all the constraints. In particular, because $\lambda R(w_1) > 1$ we have that the collateral constraint is:

$$b_1 \leq \lambda R(w_1) k_1 \Longleftrightarrow A \leq \lambda R(w_1) A$$

which is always true, because $\lambda R(w_1) > 1$. If, on the contrary, we had that $\lambda R(w_1) < 1$, then this plan would never be feasible, since $A > \lambda R(w_1) A$, violating the collateral constraint.

Case 4 $\lambda R(w_1) < 1 < R(w_1)$

We will show that in this case, the problem is well defined, and has a positive solution for capital. See that in the optimum we must have that

$$\begin{aligned} c_0 &= E_0 + b_1 - k_1 \geq 0 \\ c_1 &= R(w_1) k_1 - b_1 \geq 0 \end{aligned}$$

so we can rewrite the optimization problem as

$$\max_{k_1, b_1} \overbrace{E_0 + b_1 - k_1}^{c_0} + \overbrace{R(w_1) k_1 - b_1}^{c_1} = E_0 + \max_{k_1, b_1} [R(w_1) - 1] k_1$$

$$s.t: \begin{cases} c_0 \equiv E_0 + b_1 - k_1 \geq 0 \\ c_1 \equiv R(w_1) k_1 - b_1 \geq 0 \\ b_1 \leq \lambda R(w_1) k_1 \end{cases} \Longleftrightarrow \begin{array}{c} (1): k_1 \leq E_0 + b_1 \\ (2): b_1 \leq R(w_1) k_1 \\ (3): b_1 \leq \lambda R(w_1) k_1 \end{array}$$

See that because $R(w_1) > 1$ we must have that the entrepreneur wants to make k_1 as big as possible. Also, see that restriction (3) implies restriction (2), so then we only need to find the biggest k_1 that satisfy the constraints

$$\begin{aligned} k_1 &\leq E_0 + b_1 \\ b_1 &\leq \lambda R(w_1) k_1 \end{aligned}$$

Because b_1 gives returns of $1 < R(w_1)$, the collateral constraint will be binding (i.e. $b_1 =$

$\lambda R(w_1) k_1$). Using that in the first constraint, we get

$$k_1 = E_0 + \lambda R(w_1) k_1 \Longleftrightarrow k_1^* = \frac{E_0}{1 - \lambda R(w_1)}$$

So, the entrepreneur can finance with his own funds E_0 of capital, and borrows from outside $\frac{E_0}{1-\lambda R(w_1)} - E_0$

(c) Show that there is a cutoff $\widehat{E}$ such that if $E_0 > \widehat{E}$, the entrepreneurs can finance the first-best level of capital $k_1 = k^* = \alpha^{\frac{1}{1-\alpha}}$, and the collateral constraint is not binding.

Answer

The first best level of capital solves

$$\max_{k_1} k_1^\alpha - k_1 \Longleftrightarrow \alpha k_1^{\alpha-1} = 1 \Longleftrightarrow k_1^* = \left(\alpha^{-1}\right)^{\frac{1}{\alpha-1}} = \alpha^{\frac{1}{1-\alpha}}$$

Also, remember that

$$R(w_1) = F_K(k) = \alpha k_1^{\alpha-1}$$

so, if we are at the first best level of capital, we have that

$$R(w_1) = \alpha \left(\alpha^{\frac{1}{1-\alpha}}\right)^{\alpha-1} = 1$$

So the collateral constraint is just satisfied (but may not bind), and the entrepreneur is indifferent between all levels of investment $k_1 \in \left[0, \frac{E_0}{1-\lambda R}\right] = \left[0, \frac{E_0}{1-\lambda}\right]$

Then, $k_1^* \in \left[0, \frac{E_0}{1-\lambda}\right] \Longleftrightarrow$

$$k_1^* < \frac{E_0}{1-\lambda} \Longleftrightarrow E_0 > (1-\lambda) k_1^* = (1-\lambda) \alpha^{\frac{1}{1-\alpha}} \equiv \widehat{E}$$

(d) Show that if $E < \widehat{E}$ there is an equilibrium where the entrepreneurs are constrained and the equilibrium value of k_1 is an increasing function of E_0.

Answer

If $E_0 < \widehat{E} \Longrightarrow k_1 < k_1^* \Longrightarrow R(w_1) > 1 \Longrightarrow$ optimal k_1 to finance is $k_1 = \frac{E_0}{1-\lambda R(w_1)}$. In equilibrium $R(w_1) = \alpha k_1^{\alpha-1}$ so $\overline{k}_1$ is given by

$$\overline{k}_1 = \frac{E_0}{1 - \lambda \alpha \overline{k}_1^{\alpha-1}}$$

Using the implicit function theorem on $G(k_1, E) = \frac{E_0}{1-\lambda\alpha k^{\alpha-1}} - k_1$ we get that

$$\frac{dk_1}{dE_0} = \frac{-G_E}{G_{k_1}} = -\frac{\frac{1}{1-\lambda\alpha k_1^{\alpha-1}}}{k_1^{\alpha-2}\alpha\lambda E_0\frac{\alpha-1}{\left(k_1^{\alpha-1}\alpha\lambda-1\right)^2} - 1} = \frac{\frac{1}{1-\lambda\alpha k_1^{\alpha-1}}}{1 + k_1^{\alpha-2}\alpha\lambda E_0\frac{1-\alpha}{\left(k_1^{\alpha-1}\alpha\lambda-1\right)^2}} > 0$$

proving the result. Note that as $E_0 \longrightarrow 0 \Longrightarrow k_1(E_0) \longrightarrow 0$ as well. If $E_0 \longrightarrow 0$ then

$$\lim_{E_0 \longrightarrow 0} \frac{dk_1}{dE_0} = \frac{\frac{1}{1-\lambda\alpha 0^{\alpha-1}}}{1 + k_1^{\alpha-2}\alpha\lambda * 0 * \frac{1-\alpha}{\left(k_1^{\alpha-1}\alpha\lambda-1\right)^2}} = 1$$

(e) Suppose the consumers pay a lump-sum tax τ in period 0. The receipts from the tax are transferred directly to the entrepreneurs. Derive an expression for the expected utility of consumers and entrepreneurs as a function of τ

If $E_0 > \widehat{E}$ then utility of entrepreneurs is maximal utility $+\tau$, and for consumers is the utility of equilibrium $-\tau$

If $E_0 < \widehat{E}$, then equilibrium capital will change, from $k_1(E_0)$ to $k_1(E_0+\tau) > k_1(E_0+\tau)$. Utility for consumers will then be

$$U_C(\tau, E_0) = w_1(k_1(E_0+\tau)) - \tau = (1-\alpha)\left[\frac{1}{k_1(E_0+\tau)}\right]^{-\alpha} - \tau$$

(f) Show, analytically or by numerical example, that there is a non-monotone relation between τ and the expected utility of the consumers. In particular, if E_0 is sufficiently small, a small positive tax can increase the utility of both consumers and entrepreneurs.

Answer

Suppose $E_0 = \varepsilon$ is sufficiently small, and τ is sufficiently small as well, so that

$$U_C(\tau, \varepsilon) = (1-\alpha)\, k_1(\varepsilon+\tau)^{\alpha} - \tau$$

At $\tau = 0$, the derivative of U_C is

$$U_C'(0, \varepsilon) = \alpha(1-\alpha)\left[k_1(\varepsilon+\tau)\right]^{\alpha-1}\frac{\partial k_1}{\partial E_0} - 1 \simeq \alpha(1-\alpha)\left[k_1(\varepsilon+\tau)\right]^{\alpha-1} - 1$$

and

$$\lim_{\varepsilon \longrightarrow 0} U_C'(0, \varepsilon) = +\infty!!$$

so, a small increase in taxes from really small endowment of entrepreneurs E_0 increase utility of consumers marginally more than one to one, so they are better off by paying the tax.

If, however, τ is big enough so that

$$\varepsilon + \tau > E_0$$

Then the tax won't affect the entrepreneur's investment in k_1, because in equilibrium $R(w_1) = 1$ and capital will be at its first best level. Therefore, wages won't change in equilibrium, and the tax will just be a lump-sum transfer from consumers to entrepreneurs with no effects on equilibrium.

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