14.454 - Macroeconomics IV
Problem Set 3 Solutions

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1 Question 1 - Jacklin's Critique to Diamond-Dygvig

Take the Diamond-Dygvig model in the recitation notes, and consider Jacklin's implementation of the social optimum via a firm that pays dividends and whose shares can be sold in a spot market at $t = 1$. Suppose now that consumers can also directly invest in the long technology, without having to invest in the firm (that is, firms do not have the exclusivity of access to the projects). Show then that in this setting, the social optimum (c_1^*, c_2^*) is not implantable. In particular, show that if the firm offers a contract with dividend $D = \pi c_1^*$, a single agent may deviate by investing all his resources in the long technology at $t = 0$ and obtain higher utility (if the rest of the agents are actually investing all their income in shares from this firm).

Answer:

Look at my recitation notes on Diamond Dygvig. The solution is posted there:

" Imagine that there is a stock market for firms running the long technology open at time $t = 1$ and $t = 2$. Suppose that everyone else is following the strategy of going to the bank, and we consider a deviant consumer that is considering investing in the long technology.

If at $t = 0$ the agent invests his endowment in one of these firms. Two things can happen

- If patient, then just waits until $t = 2$ and get $R > c_2^*$

- If impatient, can sell a patient agent the right to get the dividends tomorrow. Recall that the equilibrium price of a share that pays $R(1-\pi c_1^*)$ is $(1-\pi)c_1^*$. Therefore, the price of this share, that pays $R>0$ units can be then sold at

$$\frac{(1-\pi)c_1^*}{1-\pi c_1^*} > c_1^*$$

So agent is better off by deviating if there is a stock market open at all dates"

2 Question 2 - Public Debt and Bursting Bubbles

There's an OLG economy of agents that live only two periods. Generation born at t has preferences $U\left(c_t^y, c_{t+1}^o\right) = \alpha \ln(c_t^y) + (1-\alpha)\ln\left(c_{t+1}^o\right)$ where superscript y stands for "young" and o for "old". Each agent is endowed with one unit of consumption good at birth, and has no endowment when old. Let N_t be the number of agents born at time t: we assume that

$$N_t = (1+g)\,N_{t-1}$$

where $g>0$ is the growth rate of the economy (and in particular, of the endowment). Also suppose that agents can only transfer resources from t to $t+1$ in a storage technology that pays 1 unit of consumption at $t+1$ by unit invested of consumption at t. Borrowing and lending among consumers does not exist, since old people cannot repay young people when these are old.

(a) Characterize the equilibrium allocation of the economy. Show that the allocation is not Pareto Optimal, by finding a Pareto Optimal scheme . In particular, find the "pay as you go" social insurance scheme in this economy, and show that it Pareto Dominates the equilibrium allocation.

Answer:

In equilibrium, there's no way to get resources from young to old individuals, so the only possibility is to save in the storage technology. The problem a typical consumer solves is

$$\max \alpha \ln(c_t^y) + (1-\alpha)\ln\left(c_{t+1}^o\right)$$

$$s.t:\begin{cases} c_t^y + S_t \leq 1 \\ c_{t+1}^o \leq S_t \end{cases}$$

where S_t is the savings in the storage technology. Since agents don't have any resources when old, this coincides with savings. Demands are then

$$\begin{aligned} c_t^y &= \alpha \\ c_{t+1}^o &= 1-\alpha = S_t \end{aligned}$$

and utility for the representative consumer is $U_{eq} = \alpha \ln(\alpha) + (1-\alpha)\ln(1-\alpha)$

In a social insurance scheme, the government charges a per capita tax τ_t to young generations to be able to pay old people their social insurance. At every point in time, the government must have a balanced budget:

$$\text{Taxes from young} = \text{Payments to the elderly} \Longleftrightarrow$$

$$\text{Per capita payment to the elderly} = \frac{\tau_{t+1}N_{t+1}}{N_t} = \tau_{t+1}(1+g)$$

Take a stationary social insurance is a social insurance scheme such that $\tau_t = \tau$ for all t. See that each generation pays τ when young and receive $\tau(1+g)$ when old, so the rate of return of the social security scheme is $g > 0$, so this "asset" dominates the storage technology (that pays a net interest rate of 0).

If agents had access to an asset that pays $(1+g)$ at $t+1$ per unit invested, then the optimization problem the consumer solves is

$$\max_{c_t^y, c_{t+1}^o} \alpha \ln(c_t^y) + (1-\alpha)\ln\left(c_{t+1}^o\right) \tag{1}$$

$$s.t:\begin{cases} c_t^y + S_t \leq 1 \\ c_{t+1}^o \leq (1+g)S_t \end{cases} \Longrightarrow c_t^y + \frac{1}{1+g}c_{t+1}^o \leq 1$$

Note that we imposed that the consumer does not save anything in the storage asset, since this hypothetical asset dominates the storage asset in returns, and has no risk at all. So, demands are

$$\begin{aligned} c_t^y &= \alpha \\ c_{t+1}^o &= (1-\alpha)(1+g) \end{aligned}$$

And see that the utility from this allocation is (obviously) better than the equilibrium allocation.

Can this allocation be implemented with social security? Yes, it can be done by setting per-capita taxes to

$$\tau = 1-\alpha$$

so agents when young consume $c_t^y = \alpha$ (which coincides with the optimum when savings in the storage technology are allowed) and get $c_{t+1}^o = \frac{\tau N_{t+1}}{N_t} = (1-\alpha)(1+g)$.

(b) Imagine now that there exist an infinitely lived government that can issue debt contracts with consumers. In particular, the government offers the following contract

- At $t = 0$, sell $D_0 > 0$ bonds at a price of 1 (this is a normalization).
- At $t = 1$, pay $R_0 D_0$ to bond holders (with $R_0 > 1$). To do this, the government issues D_1 bonds to pay interest, and the scheme keeps on going.

In general, the budget constraint of the government at time t is then

$$D_t - D_{t-1}R_{t-1} \geq 0$$

The government in this case is creating a new asset, that was not in the economy before (also note that this is exactly a Ponzi scheme, but run by the government!). Show that by picking a constant debt per capita $d \equiv \frac{D_t}{N_t}$ and a constant gross interest rate $R > 1$ the government can implement the same as in the Social Security Scheme

Answer:

Take a consumer born at time t, and let's get his demand functions. Note that since $R > 1$, the agent will never use the storage technology if debt pays more than 1 per dollar invested. If we can show that there exist an interest rate R and a debt supply $\{D_t\}_{t=0}^{\infty}$ such that:

$$\begin{aligned} R &= 1+g \\ D_t &= (1-\alpha) N_t \end{aligned}$$

then we can invoke the solution to (1) to argue that the equilibrium allocation will be the same as the one with a social security scheme, in which agents only save in public debt, and not in the storage technology.

If $\frac{D_t}{N_t} = d$ is a constant and $R_t = R$ for all t, then the budget constraint of the government can be rewritten as

$$D_t - D_{t-1}R_{t-1} \geq 0 \Longleftrightarrow dN_t - dN_{t-1}R \geq 0 \Longleftrightarrow d\frac{N_t}{N_{t-1}} \geq R \Longleftrightarrow$$

$$d(1+g) \geq R$$

Then, if the government sets interest rate to $R = (1+g)$, then any per capita debt level $d > 0$ is feasible from the point of view of the government. However, there must be equilibrium in the debt market:

$$d = \text{Demand of debt by generation } t \Longleftrightarrow$$

$$d = 1 - \alpha$$

So, by setting $D_t = (1-\alpha) N_t$ and a constant interest rate $R = (1+g)$, the government has the ability of creating the "missing asset" that generates the Pareto optimal interest rate of $1 + g > 1$.

(c) Imagine now that there is no public debt issued by the government, but rather that there exist a "bubble" asset B that pays no dividend. Let $\{p_t\}_{t=0}^{\infty}$ be the equilibrium prices of the bubble. Imagine too that the bubble may burst: in every history in which the bubble did not burst, the bubble burst in the next period with probability $(1-\lambda)$ where $\lambda \in (0,1)$. If the bubble bursts, then the bubble has no value from tomorrow on (i.e. $p_T = 0$ for all $T \geq t$)

Consider the consumption problem of a young agent born at time t, and that the bubble has not bursted until now. Let S_t be the savings of generation t in the storage technology, and B_t be the demand for the bubble asset. Find the optimal asset demands

$$\begin{aligned} S_t &= S\left(p_t, p_{t+1}^S\right) \\ B_t &= B\left(p_t, p_{t+1}^S\right) \end{aligned}$$

where p_{t+1}^S is the price of the bubble at $t+1$ if the bubble does not burst (if it does, then the price is 0). Do this under the assumption that $p_{t+1}^S > p_t$

Answer:

The consumer problem is now

$$\max_{c_1, c_2^C, c_2^S, S, B} \alpha \ln(c_1) + (1-\alpha)(1-\lambda)\ln\left(c_2^C\right) + (1-\alpha)\lambda \ln\left(c_2^S\right)$$

$$s.t: \begin{cases} c_1 + p_t B + S \leq 1 \\ c_2^C \leq S \\ c_2^S \leq S + p_{t+1} B \end{cases}$$

Where, to simplify notation, $c_1 = c_t^y$, $c_2^C = c_{t+1}^o$ if the bubble collapses (i.e. its price goes to 0), $c_2^S = c_{t+1}^o$ if the bubble subsists, $S =$ investment in storage asset and $B =$ net purchases of the bubble asset.

Note that

$$c_2^S = c_2^C + p_{t+1}B \Longleftrightarrow \frac{c_2^S}{c_2^C} = 1 + p_{t+1}\frac{B}{S}$$

and also, that

$$B = \frac{c_2^S - c_2^C}{p_{t+1}}, S = c_2^C$$

Solve for (S, B) :

$$\max_{S,B} \alpha \ln(1 - S - p_t B) + (1-\alpha)(1-\lambda)\ln(S) + (1-\alpha)\lambda \ln(S + p_{t+1}B)$$

$$(B) : -\alpha \frac{p_t}{1 - S - p_t B} + (1-\alpha)\frac{\lambda p_{t+1}}{S + p_{t+1}B} = 0 \Longleftrightarrow$$

$$\left(\frac{1-\alpha}{\alpha}\right)\frac{\lambda p_{t+1}}{p_t} = \frac{S + p_{t+1}B}{1 - S - p_t B} = \frac{c_2^S}{c_1} \tag{2}$$

$$(S) : \frac{1-\lambda}{S} + \frac{\lambda}{S + p_{t+1}B} = \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{1 - S - p_t B} \Longleftrightarrow$$

$$(S) : 1 - \lambda + \lambda \frac{1}{1 + p_{t+1}\frac{B}{S}} = \left(\frac{\alpha}{1-\alpha}\right)\frac{S}{1 - S - p_t B} \Longleftrightarrow$$

$$1 - \lambda + \lambda \frac{c_2^C}{c_2^S} = \frac{\alpha}{1-\alpha}\frac{c_2^C}{c_1} \Longrightarrow$$

$$1 - \lambda + \lambda \frac{\left(\frac{c_2^C}{c_1}\right)}{\left(\frac{c_2^S}{c_1}\right)} = \frac{\alpha}{1-\alpha}\frac{c_2^C}{c_1} \Longrightarrow 1 - \lambda + \frac{\alpha}{1-\alpha}\frac{p_t}{p_{t+1}}\left(\frac{c_2^C}{c_1}\right) = \frac{\alpha}{1-\alpha}\frac{c_2^C}{c_1} \Longleftrightarrow$$

$$\frac{c_2^C}{c_1}\frac{\alpha}{1-\alpha}\left(1 - \frac{p_t}{p_{t+1}}\right) = 1 - \lambda \Longleftrightarrow$$

$$\frac{c_2^C}{c_1} = \left(\frac{p_{t+1}}{p_{t+1} - p_t}\right)\left(\frac{1-\alpha}{\alpha}\right)(1-\lambda) \tag{3}$$

Putting everything into the first budget constraint we have

$$c_1 + p_t B + S = 1 \Longleftrightarrow c_1 + \frac{p_t}{p_{t+1}}\left(c_2^S - c_2^C\right) + c_2^C = 1 \Longleftrightarrow$$

$$\frac{1}{c_1} = 1 + \frac{p_t}{p_{t+1}}\left(\frac{1-\alpha}{\alpha}\right)\left[\frac{\lambda p_{t+1}}{p_t} - \left(\frac{p_{t+1}}{p_{t+1} - p_t}\right)(1-\lambda)\right] + \left(\frac{p_{t+1}}{p_{t+1} - p_t}\right)\left(\frac{1-\alpha}{\alpha}\right)(1-\lambda) \Longleftrightarrow$$

$$\frac{1}{c_1} = 1 + \frac{1-\alpha}{\alpha}\left[\lambda - \left(\frac{p_t}{p_{t+1} - p_t}\right)(1-\lambda)\right] + \left(\frac{p_{t+1}}{p_{t+1} - p_t}\right)\left(\frac{1-\alpha}{\alpha}\right)(1-\lambda) \Longleftrightarrow$$

$$\frac{1}{c_1} = \frac{1}{\alpha} \Longrightarrow$$

$$c_1 = \alpha \tag{4}$$

and then

$$c_2^C = \left(\frac{p_{t+1}}{p_{t+1} - p_t} \right) (1 - \alpha)(1 - \lambda) = S \tag{5}$$

$$c_2^S = (1 - \alpha) \frac{p_{t+1}}{p_t} \lambda \tag{6}$$

So

$$B = \frac{1}{p_{t+1}} (1 - \alpha) \frac{p_{t+1}}{p_t} \lambda - \frac{1}{p_{t+1}} \left(\frac{p_{t+1}}{p_{t+1} - p_t} \right) (1 - \alpha)(1 - \lambda) \Longleftrightarrow$$

$$B = (1 - \alpha) \frac{1}{p_t} \lambda - \left(\frac{1}{p_{t+1} - p_t} \right) (1 - \alpha)(1 - \lambda) = \left[\frac{\lambda}{p_t} - \frac{(1 - \lambda)}{p_{t+1} - p_t} \right] (1 - \alpha) \Longleftrightarrow$$

$$B = B_t = B\left(p_t, p_{t+1}^S\right) \equiv \left[\frac{\lambda}{p_t} - \frac{(1 - \lambda)}{p_{t+1}^B - p_t} \right] (1 - \alpha) \tag{7}$$

and

$$S_t = S\left(p_t, p_t^B\right) \equiv \left(\frac{p_{t+1}^B}{p_{t+1}^B - p_t} \right) (1 - \alpha)(1 - \lambda)$$

See that now, because of risk aversion and the possibility of the bubble bursting, agents want to save some in the storage technology just in case the bubble bursts. See that if the bubble never collapses (i.e. $\lambda = 1$) then

$$\begin{aligned} p_t B &= 1 - \alpha \\ S &= 0 \end{aligned} \tag{8}$$

which is the case we saw in class: savings are replicated by investing in the bubble, which amounts to resources $p_t B = 1 - \alpha$, and pays $\frac{p_{t+1}^B}{p_t}(1 - \alpha)$ if the bubble does not crash.

(d) Find a stationary equilibrium with bubbles in this economy, in which

$$\frac{p_{t+1}^B}{p_t} = 1 + \gamma > 1 \text{ for all } t: \text{ bubble did not burst at } t$$

And show that in any such equilibrium, $\gamma = g$. Does this equilibrium implement the "pay as you go" social insurance allocation? If it does not: which one is better?

Answer:

Equilibrium in the bubble asset market implies that:

$$\text{supply of generation born at } t-1 = \text{ demand of generation born at } t$$

$$\left[\frac{\lambda}{p_{t-1}} - \frac{(1-\lambda)}{p_t - p_{t-1}}\right](1-\alpha)\,N_{t-1} = \left[\frac{\lambda}{p_t} - \frac{(1-\lambda)}{p_{t+1} - p_t}\right](1-\alpha)\,N_t$$

Since $N_t = (1+g)\,N_{t-1} \Longrightarrow$

$$\left[\frac{\lambda}{p_{t-1}} - \frac{(1-\lambda)}{p_t - p_{t-1}}\right] = \left[\frac{\lambda}{p_t} - \frac{(1-\lambda)}{p_{t+1} - p_t}\right](1+g)$$

And take a stationary equilibrium in which

$$\frac{p_{t+1}}{p_t} = \frac{p_t}{p_{t-1}} = 1+\gamma$$

Then

$$\left[\frac{\lambda}{p_{t-1}} - \frac{(1-\lambda)}{\left(\frac{p_t}{p_{t-1}} - 1\right)p_{t-1}}\right] = \left[\frac{\lambda}{p_t} - \frac{(1-\lambda)}{\left(\frac{p_{t+1}}{p_t} - 1\right)p_t}\right](1+g) \Longleftrightarrow$$

$$\lambda - \frac{1-\lambda}{\frac{p_t}{p_{t-1}} - 1} = \left[\lambda\frac{p_{t-1}}{p_t} - \frac{(1-\lambda)}{\left(\frac{p_{t+1}}{p_t} - 1\right)}\frac{p_{t-1}}{p_t}\right](1+g) \Longleftrightarrow$$

$$\lambda - \frac{1-\lambda}{\gamma} = \left[\frac{\lambda}{1+\gamma} - \frac{(1-\lambda)}{\gamma}\left(\frac{1}{1+\gamma}\right)\right](1+g) \Longleftrightarrow$$

$$\lambda - \frac{1-\lambda}{\gamma} = \left[\lambda - \frac{(1-\lambda)}{\gamma}\right]\frac{1}{1+\gamma}(1+g) \Longleftrightarrow$$

$$\gamma = g$$

So, the equilibrium demand for bubbles and for the storage asset are

$$p_t B = \left[\lambda - \frac{(1-\lambda)}{g}\right](1-\alpha) \tag{9}$$

and

$$S_t = \left(1 + \frac{1}{g}\right)(1-\alpha)(1-\lambda)$$

See that the higher the prob. of survival, and the higher g, then the higher the investment in the bubble. Of course, this allocation does not implement the pareto optimal allocation of social security, since the investment in the storage technology is positive. This is because the fragility of the bubble makes agents try to save for precautionary motives in the "risk free" asset, that pays a lower interest rate. The equilibrium allocation is only efficient whenever $\lambda = 1$, and is inefficient if $\lambda < 1$. In particular, is easy to see that expected utility for the representative consumer is increasing in λ, so a social planner would impose $\lambda = 1$ in the optimum (by "forcing" coordination).

(e) Suppose now that on top of the bubble, the government issues the debt contract specified in part (b). Will there be bubbles in equilibrium? Think about policy implications of your findings

Answer:

If the government can (believably) promise a payment of $R = (1+g) > 0$ to all generations holding $d = (1-\alpha)$, then in an equilibrium where debt has value (Agents demand the debt) and such that $\lambda < 1 \Longrightarrow$ demand for the bubble will be zero and $p_t = 0$ for all t. The reason is really easy: the expected return of these competing assets are

$$\text{expected return of bubble} = \lambda(1+g) < (1+g) = \text{expected return debt}$$

Moreover, debt has no risk in this setting (this is certainly not true in general!! Take the case of Argentina...) so debt clearly dominates bubble assets as an asset in which to store value. Therefore, the demand for the bubble by consumers in a steady state equilibrium will be zero, consistent with a price of 0.A clear policy implication for this is that the government could compete with the bubble asset by creating an asset with the same rate of return of the bubble, but with a smaller risk of it collapsing (in this model, there is absolutely no risk in public debt).

3 Question 3 - Bubbles and Investment

Consider an economy with 2−period lived overlapping generations of agents. Population is constant. When young, agents have a unit endowment of labor, which they supply inelastically on the labor market at the wage w_t. They consume $c_{t,t}$ and save $w_t - c_{t,t}$. For the moment, assume all their savings go into physical capital k_{t+1}. When old, they rent capital at the rate r_t and consume $c_{t,t+1} = r_t k_t$. Their preferences are

$$\ln(c_{t,t}) + \beta \ln(c_{t,t+1})$$

The production function is Cobb-Douglas:

$$y_t = k_t^{\alpha} l_t^{1-\alpha}$$

(a) Solve the optimal savings problem of the consumer born at time t, taking as given the prices w_t and r_t

Answer:

Note: These solutions are from Alp Simsek's solutions
Young agents of generation t solve:

$$\max_{c_{t,t}, c_{t,t+1}} \ln(c_{t,t}) + \beta \ln(c_{t,t+1})$$

$$s.t: \begin{cases} c_{t,t} + k_{t+1} \leq w_t \\ c_{t,t+1} \leq r_{t+1} k_{t+1} \end{cases}$$

The fact that the budget constraints are binding implies $c_{t,t} = w_t - k_{t+1}$ and $c_{t,t+1} = r_{t+1} k_{t+1}$. Plugging these expressions into the objective function and considering the first order condition implies:

$$\frac{1}{w_t - k_{t+1}} = \frac{\beta}{k_{t+1}}$$

Note that the rental rate of capital, r_{t+1} drops out of the first order condition, which is a feature of log utility (income and substitution effects cancel). Using this first order condition, the optimal consumption and investment plan is solved as:

$$k_{t+1} = \frac{\beta}{1+\beta} w_t \text{ and } c_{t,t} = \frac{1}{1+\beta} w_t \tag{10}$$

That is, consumers save a constant fraction of their income and their saving rate is increasing in their discount factor, β

(b) Solve the problem of the representative firm and use market clearing in the labor market to derive expressions for w_t and r_t as functions of k_t

Answer:

The representative firm chooses labor and capital inputs to maximize:

$$\max_{k_t, l_t} k_t^{\alpha} l_t^{1-\alpha} - r_t k_t - w_t l_t$$

Using the first order condition for this problem, along with factor market clearing (in particular, $l_t = 1$ for all t), the equilibrium factor returns are given by:

$$\begin{aligned} r_t &= \alpha k_t^{\alpha-1} \\ w_t &= (1-\alpha) k_t^{\alpha} \end{aligned} \tag{11}$$

(c) Substitute the result in (b) in the optimal savings rule derived in (a) and obtain a law of motion for k_t

Answer:

Substituting the expression for the wages into the saving rule in (10), the law of motion of capital is given by:

$$k_{t+1} = \frac{\beta}{1+\beta} (1-\alpha) k_t^{\alpha}$$

(d) Find a steady state with constant capital stock $k_t = k_{SS}$. Show that if

$$\frac{\alpha}{1-\alpha} \frac{1+\beta}{\beta} < 1 \tag{12}$$

then $\alpha k_{SS}^{\alpha-1} < 1$

Answer:

The steady state capital level, k_{ss} is found by setting $k_{t+1} = k_t = k_{ss}$ in the law of motion for capital. This gives

$$k_{ss} = \left[\frac{\beta (1-\alpha)}{1+\beta} \right]^{\frac{1}{1-\alpha}}$$

From the factor price expression (11), the steady-state rental rate of capital is given by:

$$r_{ss} = \alpha k_{ss}^{\alpha-1} = \frac{\alpha}{1-\alpha} \frac{1+\beta}{\beta}$$

If the parameters are such that the right hand side expression is less than 1, then the steady-state rental rate of capital is given by $r_{ss} < 1$

(e) Suppose the economy begins at $t = 0$ at the steady state capital stock. Write down the resource constraint of the economy in steady state and argue that if $\alpha k_{SS}^{\alpha-1} < 1$ it is possible to make all agents better off by reducing the capital stock in all periods.

Answer:

Note that since there is no population growth, the resource constraint of the economy is given by:

$$c_{t,t} + c_{t-1,t} \leq k_t^{\alpha} - k_{t+1} \text{ for all } t$$

In view of this resource constraint, consider the steady-state level of capital that maximizes net output, $k^{\alpha} - k$. This level, also known as the golden rule level of capital, is the unique solution to:

$$\alpha k_{gr}^{\alpha-1} = 1$$

Since $r_{ss} = \alpha k_{ss}^{\alpha-1} < 1$, we have $k_{ss} > k_{gr}$, that is, the steady-state capital level is above the golden rule level.

Consider next a reduction of the steady-state capital level to the golden rule level; i.e. consider the capital allocation

$$k_t = k_{gr} < k_{ss} \text{ for each } t \geq 1 \tag{13}$$

This allocation is feasible and increases the net output at all dates $t \geq 1$. To understand the effect of this allocation on the net output at date 0, consider the resource constraint at date 0:

$$c_{0,0} + c_{-1,0} \leq k_0^{\alpha} - k_1 = k_{ss}^{\alpha} - k_1$$

At the equilibrium allocation, $k_1 = k_{ss}$, which implies that the net output at date 0 is $k_{ss}^{\alpha} - k_{ss}$. At the proposed allocation in (13) which implies that the net output at date 0 is given by $k_{ss}^{\alpha} - k_{gr} > k_{ss}^{\alpha} - k_{ss}$. Hence, the proposed allocation in (13) increases the net output at all dates, including date 0. Note also that the increased net output can be distributed across the agents in a way to make everyone better off (we can just give agents their equilibrium consumptions plus a distribution of the increase in net output). Consequently, by reducing the capital stock in every period, it is possible to increase the welfare of all agents in this economy, i.e. the competitive equilibrium is Pareto inefficient.

(f) Suppose now agents are allowed to trade a useless, non-reproducible asset, in fixed unit supply, which trades at the price p_t, the "bubble" asset. Argue that if $p_t > 0$ and $k_t > 0$, the agent must be indifferent between holding capital and the bubble asset, and derive the associated arbitrage condition.

Answer:

Let b_t denote the agents' holding of the bubble asset at date t (which will be equal to the fixed supply, 1, in equilibrium). Young agents of generation t solve:

$$\max_{c_{t,t},c_{t,t+1},k_{t+1}\geq 0,b_{t+1}\geq 0} \ln(c_{t,t}) + \beta \ln(c_{t,t+1})$$

$$s.t: \begin{cases} c_{t,t} + k_{t+1} + p_t b_{t+1} \leq w_t \\ c_{t,t+1} \leq r_{t+1}k_{t+1} + p_{t+1}b_{t+1} \end{cases}$$

The budget constraints are satisfied with equality, which implies $c_{t,t} = w_t - p_t b_{t+1} - k_{t+1}$ and $c_{t,t+1} = r_{t+1}k_{t+1} + p_{t+1}b_{t+1}$. Plugging these expressions into the objective function, we obtain the first order conditions:

$$\begin{aligned} \frac{1}{w_t - p_t b_{t+1} - k_{t+1}} &= \frac{r_{t+1}}{r_{t+1}k_{t+1} + p_{t+1}b_{t+1}} \\ \frac{p_t}{w_t - p_t b_{t+1} - k_{t+1}} &= \frac{p_{t+1}}{r_{t+1}k_{t+1} + p_{t+1}b_{t+1}} \end{aligned} \quad (14)$$

Combining these expressions, we obtain the no-arbitrage condition:

$$r_{t+1} = \frac{p_{t+1}}{p_t} \quad (15)$$

That is, the rate of return on the two alternative assets, capital and the bubble, should be equal to each other.

(g) Show that if (12) holds, then there exists a steady state equilibrium with $p_t = p_{SS} > 0$ and $\alpha k_{SS}^{\alpha-1} = 1$

Our goal is to substitute for the factor market clearing conditions (11) and the bubble market clearing $b_t = b_{t+1} = 1$, into the first order conditions (14) and (15) to obtain two difference equations in k_t and p_{t+1}.

First note that rearranging condition (14) gives:

$$\begin{aligned} w_t - p_t b_{t+1} - k_{t+1} &= k_{t+1} + \frac{p_{t+1}}{r_{t+1}} b_{t+1} \\ &= k_{t+1} + p_t b_{t+1} \end{aligned}$$

where the second line uses the no-arbitrage condition. After using $b_t = b_{t+1} = 1$ and rearranging terms, the last equation can be rewritten as:

$$k_{t+1} + p_t = \frac{\beta}{1+\beta} w_t \tag{16}$$

This equation is intuitive: the right hand side is the agents' total saving which only depends on w_t in view of log utility, and the left hand side is the two components of the saving (capital and bubble investments). Note that the presence of bubble "crowds out" investment in capital, which will reduce the equilibrium level of capital, as we will shortly see. Using the factor market clearing condition (11) in the last equation, we obtain a first relationship between capital and bubble price:

$$k_{t+1} + p_t = \frac{\beta}{1+\beta} (1-\alpha) k_t^{\alpha} \tag{17}$$

Next note that using the factor market clearing condition (11) in the no-arbitrage condition (15) , we obtain a second relationship

$$\alpha k_{t+1}^{\alpha-1} = \frac{p_{t+1}}{p_t} \tag{18}$$

The equilibrium path (k_t, p_t) is determined by the difference equations (17)–(18) along with the initial conditions for k_0. Note that we are missing one initial or end-value condition, which will lead to more than one solution to the equations.

To find a steady state equilibrium, we plug in $p_t = p_{ss}$ and $k_t = k_{ss}$ into (17) – (18). The second equation gives:

$$\alpha k_{ss}^{\alpha-1} = 1$$

which implies that the steady-state capital level is equal to the golden rule level. the first equation solves for the price level of the bubble:

$$p_{ss} = \frac{\beta}{1+\beta} (1-\alpha) k_{ss}^{\alpha} - k_{ss}$$

Note that this is positive if, and only if condition (12) is satisfied. Suppose this is the case. Then this analysis establishes that, when the bubbleless economy is Pareto inefficient, there exists a steady-state equilibrium in which there is positive investment in the bubble and the capital is equal to the golden rule level. As we have seen in part (c), this equilibrium has a higher net output at all dates than the bubbleless equilibrium. It can also be checked that this equilibrium is Pareto efficient. Hence, the bubble solves the Pareto inefficiency problem in this economy. Intuitively, the source of the inefficiency in this economy is over-investment. The agents have a large desire to save (note that condition (12) holds when is sufficiently large), and thus they are looking for a store of value. If they put all of their saving into productive capital, the equilibrium rental rate of return falls below 1. However, since this is an infinitely lived economy, dynamic trade between generations can generate a rate of return equal to 1. The bubble implements this dynamic trade. The young agents transfer their some of their resources to the old agents by buying the worthless asset. In turn, tomorrow's young agents transfer some of their resources to today's young agents, by buying the worthless asset from them. For a related intuition, recall Eq. (16), which shows that bubble crowds out investment in capital. Since the initial problem is over-investment in capital, crowding out capital increases the net output in all periods. Note, however, that bubble is able to crowd out investment only because bubble (by implementing the dynamic trade) is able to promise a higher rate of return, 1, than the rate of return to capital in the bubbleless equilibrium.

4 Question 4: Allen & Gale (2000) - Fundamental Values

Take the "Bubbles and Crisis" model seen in class (Lecture notes 5). We want to get the fundamental price of investing in the risky asset without risky asset.

(a) Consider the "complete contracts" setting (i.e. with no bankruptcy or default), in which a single risk neutral agent endowed with wealth $B > 0$ has to decide how much to invest in the safe asset (X_S) , and how much to invest in the risky asset (X_R) to maximize expected profits (minus non-pecuniary costs), subject to the constraint that $X_S + PX_R \leq B$. Set up the problem and set the first order condition for X_R

Answer:

The program here, abstracting from contracting problems, is then

$$\max_{X_S, X_R} \int_0^{\overline{R}} (rX_S + RX_R)\, h(R)\, dR - c(X_R)$$

$$s.t : X_S + PX_R \leq B$$

that is, is the same problem considered in class, but default is not allowed (because returns from both projects are considered contractable). The Lagrangian is then

$$\mathcal{L} = \int_0^{\overline{R}} (rX_S + RX_R)\, h(R)\, dR - c(X_R) + \lambda (B - X_S - PX_R)$$

and FOCs are

$$(X_S) : r = \lambda \tag{19}$$

$$(X_R) : \mathbb{E}(R) - c'(X_R) - P\lambda = 0 \underbrace{\iff}_{\text{using (19)}} P = \frac{1}{r}\left[\mathbb{E}(R) - c'(X_R)\right]$$

(b) Setting $X_R = 1$ and using the FOC found in (a), find P^f as the price at which an agent who invests his own money would be willing to hold one unit of the risky asset (that is, the marginal utility of having an extra unit of risky asset in the optimum plan).

Answer:

By setting $X_R = 1$ (the market clearing condition in the model) we get

$$P^f \equiv \frac{1}{r}\left[\mathbb{E}(R) - c'(X_R)\right]$$

which is the price that would clear the market of the risky asset in an economy where default was prohibited (since the first welfare theorem holds here).

(c) Under what conditions will we have that $P^f > P$, where P is the equilibrium price?

Answer:

See Lecture notes 5, slides 10-11.

5 Question 5: Caballero & Krishnamurthy (2006) - Welfare and Bubbles

Consider the model in Caballero and Krischnamurthy (JME, 2006) that we studied in class. Compute equilibrium welfare in the case where there is no bubble and in the case where there is a bubble. Compare and determine whether: **(a)** it is always better to have a bubble; **(b)** it is always better to have no bubble;

or **(c)** it depends on parameters. Throughout, you must assume the parameter restrictions imposed in the paper are satisfied.

Answer:

Let's first compute the ex-ante welfare (that is, as generation t consumers compute it at time t) in the case where there is no bubbly asset available. In this case, following the analysis for Caballero and Krishnamurty (2006), under the assumption that $W_t = K_t$ (that is, before-investment in domestic goods international goods are the same) and that $\psi R < 1$ (assumption we will maintain throughout this analysis) we had that the price in the loans market at $t+1$ is $p_{t+1} = 1$ for all t. This implies that the profits of the banker that lends l_{t+1} plus the consumption she had are $c^b_{t+1} = W_t(1+r^*) + RK_t + p_{t+1}l_t - l_{t+1} = (p_{t+1}-1)l_{t+1} + W_t(1+r^*) + RK_t = W_t(1+r^*) + RK_t$. On the other hand, entrepreneurs (which are half of the population) borrow up to their collateral financial constraint, so $l_{t+1} = \frac{\psi RK_t}{p_{t+1}} = \psi RK_t$ which means that the profits they get from their technology are

$$c^e_{t+1} = R\left(W_t(1+r^*) + l_{t+1}\right) + RK_t - p_{t+1}l_{t+1}$$

and using that we assumed $W_t = K_t$ and using the equilibrium investment level and prices, we get

$$c^e_{t+1} = (R-1)\psi RK_t + RK_t(1+r^*) + RK_t$$

Denote ω^{NB}_t be the ex ante welfare of generation t. Given that with probability $\frac{1}{2}$ they are either bankers or entrepreneurs and that utility is linear, we have that

$$\omega^{NB}_t = \frac{1}{2}c^b_{t+1} + \frac{1}{2}c^e_{t+1} = \frac{1}{2}\left(K_t(1+r^*) + RK_t\right) + \frac{1}{2}\left((R-1)\psi RK_t + RK_t(1+r^*) + RK_t\right) \iff$$

$$\omega^{NB}_t = RK_t + \frac{(R+1)}{2}(1+r^*)K_t + \frac{(R-1)}{2}\psi RK_t$$

Now, let's turn to the bubble equilibrium case. For this, the authors make two extra assumptions:

$$\begin{aligned}(1) &: \quad (1-\lambda)\Delta r^b - \lambda(1+r^*) > 0 \\ (2) &: \quad (1-\lambda)\Delta r^b - \lambda(1+r^*)\frac{2R}{1+R} < 0\end{aligned}$$

with $\Delta r^b \equiv g - r^* > 0$. With this two assumptions, the authors prove that if we define p^B_{t+1} the price of loans if the bubble does not crash, p^C_{t+1} if the bubble doe not crash and α_p the proportion of assets invested in the bubbly asset, we have that $p_{t+1}^B = 1$ and $p_{t+1}^C = p^C$ with p^C and α_p satisfying:

$$\begin{aligned}(1-\lambda)\frac{\Delta r^b}{1+r^*}(R+1) - \lambda\left(R+p^C\right) &= 0 \\ p^C &= \frac{\psi R}{(1+r^*)(1-\alpha_p)}\end{aligned}$$

which yields $p^C = \Delta r^b \left(\frac{1-\lambda}{\lambda}\right)\left(\frac{1+R}{1+r^*}\right) - R \in [1, R]$ under our assumptions and

$$\alpha_p = 1 - \frac{\psi R}{(1+r^*)\left(\Delta r^b \left(\frac{1-\lambda}{\lambda}\right)\left(\frac{1+R}{1+r^*}\right) - R\right)} \in (0,1)$$

Then, denoting $c_{t+1}^{i,j}$ be consumption of type i consumer ($i = b, e$) in state j ($j = C, B$), following the same steps as we did when calculating the no bubble equilibrium, we have

$$\omega_{t+1}^B = (1-\lambda)\left(\frac{1}{2}c_{t+1}^{b,B} + \frac{1}{2}c_{t+1}^{e,B}\right) + \lambda\left(\frac{1}{2}c_{t+1}^{b,C} + \frac{1}{2}c_{t+1}^{e,C}\right) =$$

$$(1-\lambda+\lambda)RK_t + (1-\lambda)\left(\frac{(R+1)}{2}\left(1+r^*+\alpha_p\Delta r^b\right)K_t + \frac{(R-1)}{2}\psi RK_t\right) + \lambda\left(\frac{\left(R+p^C\right)}{2}(1+r^*)(1-\alpha_p)K_t + \frac{\left(R-p^C\right)}{2}\left(\frac{\psi RK_t}{p^C}\right)\right)$$

which can be rewritten as

$$\begin{aligned}\omega_{t+1}^B &= RK_t + \frac{1}{2}\alpha_p\left((1-\lambda)\frac{\Delta r^b}{1+r^*}(R+1) - \lambda\left(R+p^C\right)\right)K_t + \\ &\quad (1-\lambda)\left(\frac{(R+1)}{2}(1+r^*)K_t + \frac{(R-1)}{2}\psi RK_t\right) + \\ &\quad \lambda\left(\frac{\left(R+p^C\right)}{2}(1+r^*)K_t + \frac{\left(R-p^C\right)}{2}\left(\frac{\psi RK_t}{p^C}\right)\right)\end{aligned}$$

and from the equations of p^C, we know that $(1-\lambda)\frac{\Delta r^b}{1+r^*}(R+1) - \lambda\left(R+p^C\right) = 0$ and $\frac{\psi RK_t}{p^C} = (1+r^*)(1-\alpha_p)K_t$, so plugging this in

$$\omega_{t+1}^{B} = RK_t + (1-\lambda)\left(\frac{(R+1)}{2}(1+r^*)K_t + \frac{(R-1)}{2}\psi RK_t\right) + \lambda\frac{RK_t}{2}(1+r^*)\left(R+p^C+\left(R-p^C\right)(1-\alpha_p)\right)$$

We want to investigate wether $\omega_{t+1}^{NB} \geq \omega_{t+1}^{B}$: this holds iff

$$RK_t + \frac{(R+1)}{2}(1+r^*)K_t + \frac{(R-1)}{2}\psi RK_t \geq RK_t +$$

$$(1-\lambda)\left(\frac{(R+1)}{2}(1+r^*)K_t + \frac{(R-1)}{2}\psi RK_t\right) + \lambda\left(\frac{\left(R+p^C\right)}{2}(1+r^*)K_t + \frac{\left(R-p^C\right)}{2}\left(\frac{\psi RK_t}{p^C}\right)\right) \iff$$

$$\lambda\left(\frac{(R+1)}{2}(1+r^*)K_t + \frac{(R-1)}{2}\psi RK_t\right) \geq \lambda\left(\frac{\left(R+p^C\right)}{2}(1+r^*)K_t + \frac{\left(R-p^C\right)}{2}\left(\frac{\psi RK_t}{p^C}\right)\right) \iff$$

$$(R+1)(1+r^*) + (R-1)\psi R \geq \left(R+p^C\right)(1+r^*) + \left(R-p^C\right)\left(\frac{\psi R}{p^C}\right) \iff$$

$$(1+r^*)\left(R+1-R-p^C\right) + \psi R\left(R-1-\frac{R}{p^C}+1\right) \geq 0 \iff$$

$$(1+r^*)\left(1-p^C\right) - \psi R^2\left(\frac{1-p^C}{p^C}\right) \geq 0 \iff$$

$$(1+r^*)\left(1-p^C\right) \geq \psi R^2\left(\frac{1-p^C}{p^C}\right)$$

Since we must have $p^C \geq 1$ given the parameter assumptions, we can further simplify to

$$(1+r^*) \leq \frac{\psi R^2}{p^C} \iff p^C \leq \frac{\psi R^2}{(1+r^*)}$$

Since $p^C = \Delta r^b\left(\frac{1-\lambda}{\lambda}\right)\left(\frac{1+R}{1+r^*}\right) - R$, we can write this entirely in terms of parameter values

$$\omega_{t+1}^{NB} \geq \omega_{t+1}^{B} \iff \Delta r^b\left(\frac{1-\lambda}{\lambda}\right)\left(\frac{1+R}{1+r^*}\right) - R \geq \frac{\psi R^2}{(1+r^*)}$$

and likewise $\omega_{t+1}^{NB} \leq \omega_{t+1}^{B} \iff \Delta r^b\left(\frac{1-\lambda}{\lambda}\right)\left(\frac{1+R}{1+r^*}\right) - R \leq \frac{\psi R^2}{(1+r^*)}$

See that condition (2) above can be rewritten as

$$\Delta r^b \left(\frac{1-\lambda}{\lambda}\right) \frac{1+R}{1+r^*} < 2R \iff \Delta r^b \left(\frac{1-\lambda}{\lambda}\right) \frac{1+R}{1+r^*} - R < R$$

So, for the bubbly equilibrium to be welfare improving, it is sufficient to satisfy the following condition:

$$R \leq \frac{\psi R^2}{(1+r^*)} \iff \psi R \geq 1+r^*$$

However, this will never hold, since $\psi R < 1$ by assumption. Therefore, we argue that we cannot use the parameter conditions to get sufficient conditions to say weather the bubbly equilibrium is better or worse than the no-bubble case. Therefore, we conjecture that it will depend on parameters. Now, look again at the restrictions, and see that we can rewrite them as

$$(1) \quad : \quad (1-\lambda)\,\Delta r^b - \lambda\,(1+r^*) > 0 \iff \frac{1-\lambda}{\lambda} > \frac{(1+r^*)}{\Delta r^b}$$

$$(2) \quad : \quad (1-\lambda)\,\Delta r^b - \lambda\,(1+r^*)\,\frac{2R}{1+R} < 0 \iff \frac{1-\lambda}{\lambda} < \frac{(1+r^*)}{\Delta r^b}\frac{2R}{1+R}$$

And define $\eta \equiv \frac{1-\lambda}{\lambda}$, so $\frac{(1+r^*)}{\Delta r^b} < \eta < \frac{(1+r^*)}{\Delta r^b}\frac{2R}{1+R}$. Then we can rewrite the condition of bubbly equilibrium to be better than no bubbles iff

$$\Delta r^b \eta \left(\frac{1+R}{1+r^*}\right) - R \geq \frac{\psi R^2}{(1+r^*)} \iff \eta \geq \frac{\psi R^2 + R\,(1+r^*)}{(1+R)\,\Delta r^b}$$

See that if η is big enough (that is, close to it's upper bound), then in the limit the condition would be

$$\Delta r^b \frac{(1+r^*)}{\Delta r^b}\frac{2R}{1+R}\left(\frac{1+R}{1+r^*}\right) - R \geq \frac{\psi R^2}{(1+r^*)} \iff 1 \geq \frac{\psi R}{(1+r^*)}$$

which is true, so as η approaches $\frac{(1+r^*)}{\Delta r^b}\frac{2R}{1+R}$ (that is, λ is low enough), then the bubbly equilibrium is welfare improving. On the other hand, as $\eta \to \frac{(1+r^*)}{\Delta r^b}$, we have that

$$\Delta r^b \frac{(1+r^*)}{\Delta r^b}\left(\frac{1+R}{1+r^*}\right) - R \geq \frac{\psi R^2}{(1+r^*)} \iff \frac{1+r^*}{R} \geq \psi R$$

which does not necessarily hold (see that for any set of parameters except for λ, we can choose λ to satisfy both conditions) if ψR is big enough. This happens when λ is relatively big enough. This clearly makes sense: as the probability of crushing increases, having a bubbly equilibrium becomes less and less desirable, since the when crash it is better to have invested simply in foreign goods.

6 Question 6: Stock Prices, Dividends and Bubbles (Exam question, 2004)

Assume you are in an economy where the stock price p_t is given by the standard arbitrage equation

$$p_t = \frac{1}{1+r}\mathbb{E}_t\left(p_{t+1} + d_{t+1}\right) \tag{20}$$

where

$$d_t - \overline{d} = \rho\left(d_{t-1} - \overline{d}\right) + \varepsilon_t \text{ where } \varepsilon_t \sim_{i.i.d} f(\varepsilon) \text{ where } \mathbb{E}_{t-1}(\varepsilon_t) = 0$$

(a) Use iterated expectations to solve for the price p_t as a function of ONLY future expected dividends. What assumption do you implicitly need to do this?

Answer:

In (20), let's substitute p_{t+1} for its expression according to (20), which is $p_{t+1} = \frac{1}{1+r}\mathbb{E}_{t+1}\left(p_{t+2} + d_{t+2}\right)$. We then get:

$$p_t = \frac{1}{1+r}\mathbb{E}_t\left(d_{t+1} + \frac{1}{1+r}\mathbb{E}_{t+1}\left(p_{t+2} + d_{t+2}\right)\right) =$$

$$\frac{1}{1+r}\mathbb{E}_t\left(d_{t+1}\right) + \left(\frac{1}{1+r}\right)^2 \mathbb{E}_t\left(\mathbb{E}_{t+1}\left(p_{t+2} + d_{t+2}\right)\right) \underset{\text{using iterated expectations}}{=}$$

$$\frac{1}{1+r}\mathbb{E}_t\left(d_{t+1}\right) + \left(\frac{1}{1+r}\right)^2 \mathbb{E}_t\left(p_{t+2} + d_{t+2}\right)$$

We can apply this result recursively I times, and we get that

$$p_t = \sum_{i=1}^{I}\left(\frac{1}{1+r}\right)^i \mathbb{E}_t\left(d_{t+i}\right) + \left(\frac{1}{1+r}\right)^I \mathbb{E}_t\left(p_{t+I}\right)$$

and taking the limit as $I \longrightarrow \infty$ we get

$$p_t = \sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i \mathbb{E}_t\left(d_{t+i}\right) + \lim_{S\longrightarrow\infty}\left(\frac{1}{1+r}\right)^I \mathbb{E}_t\left(p_{t+I}\right)$$

and it depends only on $\{d_{t+i}\}_{i=1}^{\infty}$ if and only if

$$\lim_{I\longrightarrow\infty}\left(\frac{1}{1+r}\right)^I \mathbb{E}_t\left(p_{t+i}\right) = K_t \tag{21}$$

for some sequence $\{K_t\}$ that does not depend on $\{d_{t+i}\}_{i=1}^{\infty}$. In particular, the fundamental price solution for (20) is the solution associated with $K = 0$:

$$\lim_{I \longrightarrow \infty} \left(\frac{1}{1+r}\right)^{I} \mathbb{E}_t\left(p_{t+i}\right) = 0$$

which is the usual "no ponzi" or "no-bubble" condition (which will become apparent later).

so we get

$$p_t = p_t^* \equiv \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^{i} \mathbb{E}_t\left(d_{t+i}\right) \tag{22}$$

which is called the "fundamental value" or "fundamental price" of the asset, which is just the expected net present value of future dividends that the asset pays off.

(b) Assume that $\rho < 1 + r$. Use iterated expectations to find an expression for the expectation (as of time t) for dividends at time $t + i = \mathbb{E}_t\left(d_{t+i}\right)$ that is a function of only $\overline{d}, \rho$ and d_t

Note: There was a typo in the pset. It said $\rho < \frac{1}{1+r}$ when it actually was $\rho < 1 + r$

Answer:

Note that $d_t - \overline{d}$ follows an $AR\left(1\right)$ process with coefficient ρ. See that

$$\mathbb{E}_t\left(d_{t+1}\right) = \mathbb{E}_t\left(d_{t+1} - \overline{d}\right) + \overline{d} = \mathbb{E}_t\left\{\rho\left(d_t - \overline{d}\right) + \varepsilon_{t+1}\right\} + \overline{d} =$$

$$\underbrace{\rho\left(d_t - \overline{d}\right)}_{\text{bec. is known at time } t} + \underbrace{\mathbb{E}_t\left(\varepsilon_{t+1}\right)}_{=0} + \overline{d} = \rho\left(d_t - \overline{d}\right) + \overline{d} \tag{23}$$

And likewise, for $i = 2$ we get:

$$\mathbb{E}_t\left(d_{t+2}\right) = \mathbb{E}_t\left(d_{t+2} - \overline{d}\right) + \overline{d} \underset{\text{bec. iterated expectations}}{=} \mathbb{E}_t\left(\mathbb{E}_{t+1}\left(d_{t+2} - \overline{d}\right)\right) + \overline{d} =$$

$$\mathbb{E}_t\left(\rho\left(d_{t+1} - \overline{d}\right) + \underbrace{\mathbb{E}_{t+1}\left(\varepsilon_{t+2}\right)}_{=0}\right) + \overline{d} = \rho\mathbb{E}_t\left(d_{t+1} - \overline{d}\right) + \overline{d} \underset{\text{from (23)}}{=} \rho^2\left(d_t - \overline{d}\right) + \overline{d}$$

and, in general we can show (recursively) that

$$\mathbb{E}_t\left(d_{t+i}\right) = \rho^i\left(d_t - \overline{d}\right) + \overline{d} \tag{24}$$

(c) Use your answers from (a) and (b) to find an expression for p_t as a function of $\overline{d}, \rho$ and d_t. Call this solution the arbitrage equation (20) the "fundamental price" p_t^*

Answer:

Using (24) we get that the fundamental price is

$$p_t^* = \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i \mathbb{E}_t\left(d_{t+i}\right) \underset{\text{using (24)}}{=} \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i \left[\rho^i \left(d_t - \overline{d}\right) + \overline{d}\right] =$$

$$\sum_{i=1}^{\infty} \left(\frac{\rho}{1+r}\right)^i \left(d_t - \overline{d}\right) + \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i \overline{d} = \underbrace{\frac{\left(\frac{\rho}{1+r}\right)}{1-\left(\frac{\rho}{1+r}\right)}}_{\text{because } \rho<1+r} \left(d_t - \overline{d}\right) + \underbrace{\frac{\left(\frac{1}{1+r}\right)}{1-\left(\frac{1}{1+r}\right)}}_{\text{because } r>0} \overline{d} =$$

$$\frac{\rho}{1+r-\rho}\left(d_t - \overline{d}\right) + \frac{1}{r}\overline{d} \tag{25}$$

which is a function of $\overline{d}, \rho$ and d_t, as we wanted to show. Note that

$$\lim_{I \longrightarrow \infty} \left(\frac{1}{1+r}\right)^I \mathbb{E}_t\left(p_{t+I}\right) = \lim_{I \longrightarrow \infty} \left(\frac{1}{1+r}\right)^I \mathbb{E}_t\left(\frac{\rho}{1+r-\rho}\left(d_{t+I} - \overline{d}\right) + \frac{1}{r}\overline{d}\right) =$$

$$\left(\frac{\rho}{1+r-\rho}\right) \lim_{I \longrightarrow \infty} \left(\frac{1}{1+r}\right)^I \mathbb{E}_t\left(d_{t+I} - \overline{d}\right) + \underbrace{\lim_{I \longrightarrow \infty} \left(\frac{1}{1+r}\right)^I \frac{1}{r}\overline{d}}_{\longrightarrow 0 \text{ as } I \longrightarrow \infty}$$

$$\underset{\text{using (24)}}{=} \left(\frac{\rho}{1+r-\rho}\right) \lim_{I \longrightarrow \infty} \left(\frac{\rho}{1+r}\right)^I \left(d_t - \overline{d}\right)$$

$$= 0 \tag{26}$$

because we assumed that $\rho < 1+r$, which implies $\frac{\rho}{1+r} < 1 \Longrightarrow \lim_{I \longrightarrow \infty} \left(\frac{\rho}{1+r}\right)^I = 0$. This implies that the fundamental price (25) is an actual solution to the difference equation (20).

(d) Now assume that the price of the stock has a bubble component $b_t = (1+r)^t b_0$ with $b_0 > 0$. Prove that the price $p_t = p_t^* + b_t$ is also a solution to the arbitrage condition (20) and that our assumption from part (a) is no longer necessary

Answer:

We only need to show that $p_t = p_t^* + b_t$ satisfies the difference equation (20). Knowing that the fundamental price p_t^* satisfies the fundamental value equation, we get

$$p_t = \frac{1}{1+r}\mathbb{E}_t\left(p_{t+1} + d_{t+1}\right) \Longleftrightarrow$$

$$p_t^* + (1+r)^t b_0 = \frac{1}{1+r}\mathbb{E}_t\left(p_{t+1}^* + (1+r)^{t+1} b_0 + d_{t+1}\right) \Longleftrightarrow$$

$$\underbrace{\left[p_t^* - \frac{1}{1+r}\mathbb{E}_t\left(p_{t+1}^* + d_{t+1}\right)\right]}_{=0 \text{ because } p_t^* \text{ is a solution for (20)}} + (1+r)^t b_0 = \frac{1}{1+r}(1+r)^{t+1} b_0 \Longleftrightarrow$$

$$(1+r)^t b_0 = (1+r)^t b_0$$

which is obviously true. However, note that

$$\lim_{I\longrightarrow\infty}\left(\frac{1}{1+r}\right)^I \mathbb{E}_t\left(p_{t+I}\right) = \underbrace{\lim_{I\longrightarrow\infty}\left(\frac{1}{1+r}\right)^I \mathbb{E}_t\left(p_{t+I}^*\right)}_{=0 \text{ using (26)}} + \lim_{I\longrightarrow\infty}\left(\frac{1}{1+r}\right)^I (1+r)^{t+I} b_0 =$$

$$(1+r)^t b_0 \lim_{I\longrightarrow\infty}\left(\frac{1}{1+r}\right)^I (1+r)^I = (1+r)^t b_0 \neq 0$$

so the assumption we did in (a) is no longer necessary. When there is bubbles, the limit $\left(\frac{1}{1+r}\right)^I \mathbb{E}_t\left(p_{t+I}\right) = (1+r)^t b_0 = K_t$ as seen in condition (21).

(e) Why are individuals willing to pay a higher price p_t for the stock than the fundamental price corresponding to the present value of the dividends, p_t^*?