

**Problem Set 2**  
 (due April 23)

**Problem 1** (Human Capital and Incomplete Markets)

We introduce human capital in the model of Angeletos and Calvet (2002) that we discussed in class. Households can now invest in two types of capital: Physical capital, denoted by  $k$  at the individual level and  $K$  at the aggregate; and human capital, denoted by  $h$  at the individual level and  $H$  at the aggregate. For simplicity, we assume that there are no financial assets other than the riskless bond; that there is no exogenous endowment; and that preferences have expected-utility representation. There are a continuum of households. The typical household faces the following problem:

$$\max E \sum_{t=0}^{\infty} \beta^t U(c_t) \quad (1)$$

$$s.t. \quad c_t + \theta_{t+1} + k_{t+1} + h_{t+1} = w_t \quad (2)$$

$$w_t = \tilde{A}_t k_t^\alpha + \tilde{B}_t h_t^\gamma + R\theta_t \quad (3)$$

where  $k$  denotes investment in physical capital,  $h$  denotes investment in human capital,  $\theta$  denotes investment in the riskless bond,  $\tilde{A}$  denotes the productivity of physical capital,  $\tilde{B}$  denotes the productivity of human capital, and  $R$  denotes the (constant) riskless rate. The utility is CARA,

$$U(c) = -\frac{1}{\Gamma} \exp(-\Gamma c).$$

The productivity shocks  $\tilde{A}$  and  $\tilde{B}$  are jointly normally distributed and i.i.d. across time and agents, and

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \sim N \left( \begin{bmatrix} \bar{A} \\ \bar{B} \end{bmatrix}, \begin{bmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{bmatrix} \right)$$

1. Solve the consumers problem. How does investment in physical and human capital depend on the variance and covariance of the shocks?
2. In the special case that  $\sigma_{AB} = 0$  and  $\alpha = \gamma = 1/2$ , solve explicitly for the optimal  $k$  and  $h$  as functions of  $R$  and  $\sigma_A, \sigma_B$ .
3. Assume  $\sigma_{AB} = 0$ . Does the optimal  $k$  depends on  $\sigma_B$ ? Why, or why not?

4. Assume  $\sigma_{AB} = 0$ ,  $\sigma_A = 0$ ,  $\sigma_B > 0$ . What is the effect of a higher  $\sigma_B$  on the optimal  $k$  and  $h$ , given  $R$ ? Consider next the general equilibrium, where we endogenize  $R$  through the Euler condition and let  $K = k, H = h$ . Discuss what is the general equilibrium effect of  $\sigma_B$  on steady-state  $K$ ,  $H$ , and  $R$ . (For this question, intuition is enough, no maths are necessary.)

5. Assume again  $\sigma_{AB} = 0$ ,  $\sigma_A = 0$ ,  $\sigma_B > 0$ , but now let the cash on hand every period be:

$$w_t = \bar{A}k_t^\alpha H_t^\eta + \tilde{B}h_t^\gamma + R\theta_t \quad (4)$$

where  $H_t$  denotes the aggregate human capital stock. Interpret this new set-up. Characterize the optimal  $k$  and  $h$  as functions of  $\sigma_B$ ,  $R$ , and  $H$ . How does  $\sigma_B$  affect the choice of  $k$  and  $h$ , given  $R$  and  $H$ ? Consider next the general equilibrium, where we endogenize  $R$  through the Euler condition and let  $K = k, H = h$  in equilibrium. Discuss what is the general equilibrium effect of a higher  $\sigma_B$  on  $H$  and  $K$ . (For this question, intuition is enough, no maths are necessary.)

6. In the light of the above results, what is likely to be the effect of higher labor-income risks on physical-capital accumulation in the presence of human-capital externalities?

**Problem 2** (based on Kiyotaki and Moore (1997), Midterm Exam 2000)

Consider an economy with an infinity of time periods and two types of continuum of agents: farmers and gatherers. The population of farmers is equal to unity. There are also two goods: an ordinary nondurable product, fruit, and a durable productive asset, land. The total supply of land is equal to  $\bar{K}$ . The farmer has a constant returns to scale technology: he uses  $k_t$  units of date  $t$  land to produce  $a_{t+1}k_t$  units of date  $t + 1$  fruit. Each farmer is always eager to expand (due to their great enjoyment of farming), but faces the credit constraint, which implies the repayment of today's debt ( $b_t$ ) does not exceed the value of land tomorrow:

$$Rb_t \leq q_{t+1}k_t, \quad (1)$$

where  $R$  is one plus real interest rate, and  $q_t$  is land price in terms of fruit. The farmer is also subject to the flow of funds constraint, which implies his investment expenditure is financed by his output and net borrowing:

$$q_t(k_t - k_{t-1}) = a_t k_{t-1} + b_t - Rb_{t-1}. \quad (2)$$

The gatherer's production displays decreasing returns to scale; they use  $k'_t$  units of date  $t$  land to produce  $G(k'_t)$  units of date  $t + 1$  fruit. The gatherer maximizes the expected discounted consumption of fruit with discount factor  $R^{-1} < 1$ . This together with land market equilibrium, implies that the residual supply of land to the farmers satisfies:

$$q_t - \frac{1}{R}q_{t+1} = \frac{1}{R}G'(\bar{K} - k_t) = \alpha k_t^{\frac{1}{\eta}}, \quad (3)$$

where  $\eta > 0$  is the elasticity of the residual supply of land with respect to the opportunity cost of holding land, and  $\alpha$  is a positive parameter.

1. Describe the steady state equilibrium of the above economy for  $a_t = a$ .
2. Suppose that the economy is at the steady state at date  $t - 1$  for some value of  $a$ , say  $a_1$ . Suppose that the value of  $a$  unanticipatedly increases to  $a_2 > a_1$  permanently from date  $t$ . Describe the equilibrium path of the economy.

**Problem 3** (Savings with Incomplete Markets, General Exam 2001)

Consider the following OLG economy with incomplete markets. Each individual lives for two periods. Population is constant. Each generation consists of a continuum of individuals. Individuals within a generation  $t$  are indexed by  $h$ . Let  $c_t^y(h)$  and  $c_{t+1}^o(h)$  denote the consumption of an individual  $h$  born in period  $t$ , when she is young and when she is old, respectively. Preferences are given by:

$$E_t U_t(h) = \log c_t^y(h) + \beta \mathbb{E}_t[\log c_{t+1}^o(h)], \quad (1)$$

where  $\beta > 0$  and  $\mathbb{E}_t$  denotes expectations. During her youth, an individual receives a fixed endowment:  $y_t^y(h) = 1$ . Part of this endowment is consumed and part is saved for the next period. Let  $k_t(h)$  denote savings. Therefore, the budget of an individual during youth is:

$$c_t^y(h) + k_t(h) = 1. \quad (2)$$

Each individual invests her savings *either* in a safe storage technology, *or* in a risky entrepreneurial project. If the individual specializes in the storage technology, her consumption during retirement is:

$$c_{t+1}^o(h) = \delta k_t(h) \quad (3)$$

where  $\delta > 0$  is constant. If instead the individual chooses to specialize in the risky entrepreneurial project, her consumption during retirement is:

$$c_{t+1}^o(h) = \tilde{A}_{t+1}(h)k_t(h). \quad (4)$$

The return  $\tilde{A}_{t+1}(h)$  is specific to individual  $h$ . It is i.i.d. across individuals, i.i.d. across time, and distributed as follows:

$$\tilde{A}_{t+1}(h) = \tilde{A} = \begin{cases} A + \sigma & \text{with probability } \frac{1}{2} \\ A - \sigma & \text{with probability } \frac{1}{2} \end{cases} \quad (5)$$

where  $A = \mathbb{E}[\tilde{A}] > \delta > 0$ ; and  $\sigma \geq 0$ . Finally, define  $g_t$  as the (gross) growth rate in the consumption of generation  $t$  over her life cycle:

$$g_t \equiv \frac{\int c_{t+1}^o(h) dh}{\int c_t^y(h) dh}. \quad (6)$$

1. Use (1)–(4) to express  $\mathbb{E}_t U_t(h)$  as a function of  $k_t(h)$ , depending on whether  $h$  specializes in storage or the risky technology.
2. Define  $\hat{\sigma}$  such that  $\mathbb{E}(\log \tilde{A}) = \log \delta$  at  $\sigma = \hat{\sigma}$ . What are the signs of  $\frac{\partial \hat{\sigma}}{\partial \delta}$  and  $\frac{\partial \hat{\sigma}}{\partial A}$ ? Next, define  $B = \exp \left[ \mathbb{E} \left( \log \tilde{A} \right) \right]$ . What are the signs of  $\frac{\partial B}{\partial \sigma}$  and  $\frac{\partial B}{\partial A}$ ? Specify the values of  $B$  at  $\sigma = 0$ , at  $\sigma = \hat{\sigma}$ , and at  $\sigma = A$ ; and draw a picture of  $B$  as  $\sigma$  varies in  $(0, A)$ . What is the interpretation of  $B$  and  $\hat{\sigma}$ ? When does individual  $h$  specialize in the risky technology, and when in storage?
3. Suppose we allow a risk-free bond to be traded. Let then  $R$  denote the risk-free rate, What is  $R$  when  $\sigma < \hat{\sigma}$ ? What is  $R$  when  $\sigma > \hat{\sigma}$ ?
4. What are the optimal savings,  $k_t(h)$ , for individual  $h$ ? How does the saving rate depend on  $R$ ,  $B$ , and  $\sigma$ ?
5. Derive  $g$  from (6). What is  $g$  at  $\sigma = 0$ ? What is  $g$  at  $\sigma < \hat{\sigma}$  and at  $\sigma > \hat{\sigma}$ ?
6. Drop (1) and suppose instead that preferences are given by

$$(c_t^y)^{1-\theta} + \beta \left\{ \mathbb{E}_t \left[ (c_{t+1}^o)^{1-\gamma} \right] \right\}^{\frac{1-\theta}{1-\gamma}}. \quad (7)$$

Then, redefine  $B$  as:

$$B(\sigma) \equiv \left\{ \mathbb{E} \left[ (\tilde{A})^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}};$$

and redefine  $\hat{\sigma}$  by  $B(\hat{\sigma}) = \delta$ . Explain the following:

- (a) What is the degree of relative risk aversion and the elasticity of intertemporal substitution implied by (7)?
- (b) How do the sign and the magnitude of  $\frac{\partial B}{\partial \sigma}$  depend on  $\gamma$ ?
- (c) What are the signs of  $\frac{\partial B}{\partial \gamma}$  and  $\frac{\partial \hat{\sigma}}{\partial \gamma}$ ?
- (d) How does the interest-sensitivity of the saving rate depend on  $\theta$ ?
- (e) How do the sign and the magnitude of  $\frac{\partial g}{\partial \sigma}$  depend on  $\gamma$  and  $\theta$ ?

**Problem 4** (Government and Growth in the Ramsey Model)

Consider a representative consumer who maximizes

$$\max \int_0^\infty e^{-\rho t} \frac{c^{1-\theta} - 1}{1-\theta} dt \quad (1)$$

subject to

$$\dot{a} = (1 - \tau_K)ra + (1 - \tau_L)w - c + v \quad (2)$$

where  $c$  denotes consumption,  $a$  denotes assets,  $r$  is the interest rate,  $w$  is the wage rate,  $\tau_K$  is the tax rate on capital income,  $\tau_L$  the tax rate on labor income, and  $v$  a lump-sum per capita transfer. The government spends  $g$  per capita in order to blow up Pacific islands (i.e.  $g$  does not affect utility or production). The government budget is

$$g + v = \tau_K r a + \tau_L w$$

The market clearing for assets is

$$k = a$$

The production function is Cobb-Douglas,  $y = k^\alpha$ , implying  $r = \alpha k^{\alpha-1}$  and  $w = (1 - \alpha)k^\alpha$ .

1. Write down the resource constraint of the economy.
2. Write down the FOCs for maximization of the consumer, taking all fiscal policy variables as given.
3. Use a phase diagram in  $(k, c)$  to show how the paths of  $k$  and  $c$  change when the government surprises people by permanently raising the values of  $\tau_K$  and  $g$ . What happens to the steady state value of  $k$ ?
4. Redo part c.) for the case in which the government raises  $\tau_L$  and  $g$  (without changing  $\tau_K$ ). What happens to the steady state value of  $k$ ? Explain the differences from those of part c.)
5. Redo part c.) for the case in which the government raises  $\tau_L$  and  $v$  (without changing  $\tau_Y$  and  $g$ ). What happens to the steady state value of  $k$ ? Explain the differences from those of parts c.) and d.)
6. Finally, assume that  $\tau_L = v = 0$  so that  $g = \tau_K y$ . Discuss the adjustment dynamics due to the following change in fiscal policy: at time  $t = T_1$ , the government announces that spending increases from  $g = 0$  to  $g = \bar{g} > 0$  until  $t = T_2$ .
7. Redo part f.), but now assuming the following policy change: at time  $t = T_1$ , the government announces that from  $t = T_2 > T_1$  until  $t = T_3 > T_2$  spending increases from  $g = 0$  to  $g = \bar{g} > 0$ . What is different?
8. Discuss how your answers to all above parts would change if labor supply was endogenous.