14.462 Advanced Macroeconomics
Spring 2004

Problem Set 2 Solution

1. First let's ignore that some goods are non-traded. The first order condition of the consumer problem is

$$\left(\int_0^N c_l^\alpha dl\right)^{\frac{1}{\alpha}-1} c_k^{\alpha-1} = \lambda p_k$$

where λ is the multiplier on the budget constraint. Solving for c_k and substituting into the budget constraint to solve for λ, we get the demand functions

$$c_k = \frac{p_k^{-\sigma}}{\bar{p}^{1-\sigma}} Y$$

where Y is income and $\bar{p} = \left(\int_0^N p_l^{1-\sigma} dk\right)^{\frac{1}{1-\sigma}}$ is the consumption based price index. Taking into account non-tradability, we get

$$c_k = \begin{cases} \frac{Y_1}{\bar{p}_1^{1-\sigma}} p_k^{-\sigma}, & k \in [0, P_1] \\ \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right] p_k^{-\sigma}, & k \in [P_1, P_1 + N_1 + N_2] \\ \frac{Y_2}{\bar{p}_2^{1-\sigma}} p_k^{-\sigma}, & k \in [P_1 + N_1 + N_2, N] \end{cases}$$

2. Profits up to a constant are given by $\left(p_k - \frac{w_i}{q_k}\right) p_k^{-\sigma}$, so profit maximization implies $p_k = \mu \frac{w_i}{q_k}$. Then output is given by

$$y_k = \begin{cases} \frac{Y_1}{\bar{p}_1^{1-\sigma}} \left(\mu \frac{w_1}{q_k}\right)^{-\sigma}, & k \in [0, P_1] \\ \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right] \left(\mu \frac{w_1}{q_k}\right)^{-\sigma}, & k \in [P_1, P_1 + N_1] \\ \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right] \left(\mu \frac{w_2}{q_k}\right)^{-\sigma}, & k \in [P_1 + N_1, P_1 + N_1 + N_2] \\ \frac{Y_2}{\bar{p}_2^{1-\sigma}} \left(\mu \frac{w_2}{q_k}\right)^{-\sigma}, & k \in [P_1 + N_1 + N_2, N] \end{cases}$$

and employment is

$$l_k = \begin{cases} \frac{Y_1}{\bar{p}_1^{1-\sigma}} \frac{(\mu w_1)^{-\sigma}}{q_k^{1-\sigma}}, & k \in [0, P_1] \\ \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right] \frac{(\mu w_1)^{-\sigma}}{q_k^{1-\sigma}}, & k \in [P_1, P_1 + N_1] \\ \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right] \frac{(\mu w_2)^{-\sigma}}{q_k^{1-\sigma}}, & k \in [P_1 + N_1, P_1 + N_1 + N_2] \\ \frac{Y_2}{\bar{p}_2^{1-\sigma}} \frac{(\mu w_2)^{-\sigma}}{q_k^{1-\sigma}}, & k \in [P_1 + N_1 + N_2, N] \end{cases}$$

3. Profits functions are

$$\pi_{P1}(q) = \frac{Y_1}{\bar{p}_1^{1-\sigma}}(\mu-1)\mu^{-\sigma}\left(\frac{w_1}{q_k}\right)^{1-\sigma}$$
$$\pi_{N1}(q) = \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right](\mu-1)\mu^{-\sigma}\left(\frac{w_1}{q_k}\right)^{1-\sigma}$$
$$\pi_{N2}(q) = \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right](\mu-1)\mu^{-\sigma}\left(\frac{w_2}{q_k}\right)^{1-\sigma}$$
$$\pi_{P2}(q) = \frac{Y_2}{\bar{p}_2^{1-\sigma}}(\mu-1)\mu^{-\sigma}\left(\frac{w_2}{q_k}\right)^{1-\sigma}$$

4. There is symmetry among exporting firms in country i. Thus different types of managerial quality used in equilibrium must give rise to the same profits. The same is true for non-exporting firms.

5. Suppose to the contrary that a manager with $q' > q$ is hired by a non-exporting firm. Then we get

$$\pi_{Pi}(q') - \omega_i(q') \geq \pi_{Pi}(q) - \omega_i(q)$$
$$\pi_{Ni}(q) - \omega_i(q) \geq \pi_{Ni}(q') - \omega_i(q')$$

   The first inequality states that non-exporting firms do at least as well with a manager of ability q' as with a manager of ability q. The second inequality states that the reverse is true for exporting firms. Combining these inequalities yields

$$\pi_{Pi}(q') - \pi_{Pi}(q) \geq \pi_{Ni}(q') - \pi_{Ni}(q).$$

   This is a contradiction: since exporting firms have a larger market, an increase in managerial ability is associated with a larger increase in profits than for non-exporting firms.

6. Each exporting firm needs a manager, so the upper critical value q_{Ni} satisfies $L_i(1 - F(q_{Ni})) = N_i$, which implies

$$q_{Ni} = F^{-1}\left(1 - \frac{N_i}{L_i}\right)$$

   Each non-exporting firm also needs a manager, so the lower critical value q_{Pi} satisfies $L_i(1 - F(q_{Pi})) = N_i + P_i$. Thus

$$q_{Pi} = F^{-1}\left(1 - \frac{N_i + P_i}{L_i}\right)$$

The price indices must satisfy

$$
\begin{aligned}
\bar{p}_1^{1-\sigma} &= L_1 \int_{q_{P1}}^{q_{N1}} \left(\mu \frac{w_1}{q}\right)^{1-\sigma} f_1(q)dq + L - 1 \int_{q_{N1}}^{\bar{q}_1} \left(\mu \frac{w_1}{q}\right)^{1-\sigma} f_1(q)dq \\
&+ L_2 \int_{q_{N2}}^{\bar{q}_2} \left(\mu \frac{w_2}{q}\right)^{1-\sigma} f_2(q)dq \\
\bar{p}_2^{1-\sigma} &= L_2 \int_{q_{P2}}^{q_{N2}} \left(\mu \frac{w_2}{q}\right)^{1-\sigma} f_2(q)dq \\
&+ L_2 \int_{q_{N2}}^{\bar{q}_2} \left(\mu \frac{w_2}{q}\right)^{1-\sigma} f_2(q)dq + L_1 \int_{q_{N1}}^{\bar{q}_1} \left(\mu \frac{w_1}{q}\right)^{1-\sigma} f_1(q)dq
\end{aligned}
$$

To see how this result is obtained, consider the first term of $\bar{p}_1^{1-\sigma}$, which corresponds to the P_1 non-exporting firms in country 1. All managers in the range $[q_{P1}, q_{N1}]$ will be assigned to this firms, and since firms are identical it does not matter which manager is assigned to which firm, so we can think of assigning them randomly. The term is then the expected value of $\left(\mu \frac{w_1}{q}\right)^{1-\sigma}$ for these goods times the number of goods:

$$
P_1 \frac{\int_{q_{P1}}^{q_{N1}} \left(\mu \frac{w_1}{q}\right)^{1-\sigma} f_1(q)dq}{\int_{q_{P1}}^{q_{N1}} f_1(q)dq}
$$

Of course by construction $\int_{q_{P1}}^{q_{N1}} f_1(q)dq = \frac{P_1}{L_1}$, so the term reduces to the one in the formula above. Another way of deriving this term is to think of managers not as assigned randomly but instead in ascending order as a function of the index of the good. Then the quality of the manger assigned to good k in country 1 is given by

$$
q_1(k) = F_1^{-1}\left(1 + \frac{k - (N_1 + P_1)}{L_1}\right)
$$

and the term can be written as

$$
\int_0^{P_1} \left(\mu \frac{w_1}{q(k)}\right)^{1-\sigma} dk.
$$

Noting that $\frac{dq}{dk} = \frac{1}{f_1(q)} \frac{1}{L_1}$, a change of variables shows that this gives the same answer.

7. It does not really matter how the schedule looks to the left of q_{Pi}, as long at it is below w_i. Part 4 pins down the shape on the intervals $[q_{Pi}, q_{Ni}]$ and $[q_{Ni}, \bar{q}_i]$. We only need to determine what happens at the critical values. Clearly it must be equal to w_i at q_{Pi}. If it is below w_i the agent would rather be a worker, if it is above w_i, then firms in the non-exporting sector would rather hire a manager with ability slightly less that q_{Pi}. The schedule must also be continuous at q_{Ni}. If it jumps up,

exporting firms would rather hire a manager with ability slightly less than q_{Ni}. If it jumps down, non-exporting firms would rather hire a manager with ability slighty above q_{Ni}. Continuity in combination with the result from part 4. implies the wage schedule for managers

$$\omega(q) = \begin{cases} w_i + \pi_{P1}(q) - \pi_{P1}(q_{Pi}) & q \in [q_{Pi}, q_{Ni}] \\ w_i + \pi_{P1}(q_{Ni}) - \pi_{P1}(q_{Pi}) + \pi_{N1}(q) - \pi_{N1}(q_{Ni}) & q \in [q_{Ni}, \bar{q}_i] \end{cases}$$

where for simplicity we set the schedule equal to w_i to the left of q_{Pi}. Taking derivatives

$$\omega'(q) = \begin{cases} \frac{Y_i}{\bar{p}_i^{1-\sigma}}(\mu-1)\mu^{-\sigma}w_i^{1-\sigma}(\sigma-1)q^{\sigma-2} & q \in (q_{Pi}, q_{Ni}) \\ \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right](\mu-1)\mu^{-\sigma}w_i^{1-\sigma}(\sigma-1)q^{\sigma-2} & q \in (q_{Ni}, \bar{q}_i) \end{cases}$$

Thus $\sigma > 2$ implies that the return to managerial quality is increasing as one moves up the quality ladder, with an upward jump at q_{Ni}.

8. Total revenue from the production of good k is given by $\mu w_i l_k$, so summing across goods yields the total income $Y_i = \mu w_i L_i F(q_{Pi}) = \mu w_i [L_i - (N_i + P_i)]$. The labor market clearing conditions are

$$L_1 \int_{q_{P1}}^{q_{N1}} \frac{Y_1}{\bar{p}_1^{1-\sigma}} \frac{(\mu w_1)^{-\sigma}}{q^{1-\sigma}} f_1(q)dq + \int_{q_{N1}}^{\bar{q}_1} L_1 \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right] \frac{(\mu w_1)^{-\sigma}}{q^{1-\sigma}} f_1(q)dq = L_1 - (N_1 + P_1)$$
$$L_2 \int_{q_{P2}}^{q_{N2}} \frac{Y_2}{\bar{p}_2^{1-\sigma}} \frac{(\mu w_2)^{-\sigma}}{q^{1-\sigma}} f_2(q)dq + L_2 \int_{q_{N2}}^{\bar{q}_2} \left[\frac{Y_2}{\bar{p}_2^{1-\sigma}} + \frac{Y_1}{\bar{p}_1^{1-\sigma}}\right] \frac{(\mu w_2)^{-\sigma}}{q^{1-\sigma}} f_2(q)dq = L_2 - (N_2 + P_2)$$

To see that one of the two is redundant, multiply the first by μw_1 and the second by μw_2:

$$L_1 \int_{q_{P1}}^{q_{N1}} \frac{Y_1}{\bar{p}_1^{1-\sigma}} \left(\frac{\mu w_1}{q}\right)^{1-\sigma} f_1(q)dq + L_1 \int_{q_{N1}}^{\bar{q}_1} \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right] \left(\frac{\mu w_1}{q}\right)^{1-\sigma} f_1(q)dq = Y_1$$
$$L_2 \int_{q_{P2}}^{q_{N2}} \frac{Y_2}{\bar{p}_2^{2-\sigma}} \left(\frac{\mu w_2}{q}\right)^{1-\sigma} f_2(q)dq + L_2 \int_{q_{N2}}^{\bar{q}_2} \left[\frac{Y_2}{\bar{p}_2^{1-\sigma}} + \frac{Y_1}{\bar{p}_1^{1-\sigma}}\right] \left(\frac{\mu w_2}{q}\right)^{1-\sigma} f_2(q)dq = Y_2$$

Now add them up and collect terms:

$$\frac{Y_1}{\bar{p}_1^{1-\sigma}} \left[L_1 \int_{q_{P1}}^{q_{N1}} \left(\frac{\mu w_1}{q}\right)^{1-\sigma} f_1(q)dq + L_1 \int_{q_{N1}}^{\bar{q}_1} \left(\frac{\mu w_1}{q}\right)^{1-\sigma} f_1(q)dq + L_2 \int_{q_{N2}}^{\bar{q}_2} \left(\frac{\mu w_2}{q}\right)^{1-\sigma} f_2(q)dq\right]$$
$$+ \frac{Y_2}{\bar{p}_2^{1-\sigma}} \left[L_2 \int_{q_{P2}}^{q_{N2}} \left(\frac{\mu w_2}{q}\right)^{1-\sigma} f_2(q)dq + L_2 \int_{q_{N2}}^{\bar{q}_2} \left(\frac{\mu w_2}{q}\right)^{1-\sigma} f_2(q)dq + L_1 \int_{q_{N1}}^{\bar{q}_1} \left(\frac{\mu w_1}{q}\right)^{1-\sigma} f_1(q)dq\right]$$
$$= Y_1 + Y_2$$

Using the formulas for the price indices, this reduces to the identity $Y_1+Y_2 = Y_1+Y_2$. The trade balance condition is

$$\frac{Y_1}{\bar{p}_1^{1-\sigma}} L_2 \int_{q_{N2}}^{\bar{q}_2} \left(\frac{\mu w_2}{q}\right)^{1-\sigma} f_2(q)dq = \frac{Y_2}{\bar{p}_2^{1-\sigma}} L_1 \int_{q_{N1}}^{\bar{q}_1} \left(\frac{\mu w_1}{q}\right)^{1-\sigma} f_1(q)dq$$

To see that it is redundant as well, combine it with the labor market clearing condition of country 1 (the version multiplied by μw_1) to obtain

$$\frac{Y_1}{\bar{p}_1^{1-\sigma}} \left[L_1 \int_{q_{P1}}^{q_{N1}} \left(\frac{\mu w_1}{q}\right)^{1-\sigma} f_1(q)dq + L_1 \int_{q_{N1}}^{\bar{q}_1} \left(\frac{\mu w_1}{q}\right)^{1-\sigma} f_1(q)dq + L_2 \int_{q_{N2}}^{\bar{q}_2} \left(\frac{\mu w_2}{q}\right)^{1-\sigma} f_2(q)dq \right]$$
$$= Y_1$$

Using the formula for $\bar{p}_1$, this reduces to the identity $Y_1 = Y_1$. Using the normalization $w_1 = 1$, any one of these three conditions can be used to determine w_2.

9. Of course the best manger $\bar{q}_1$ is the one that receives the highest wage, and using the normalization $w_1 = 1$:

$$\begin{aligned}\omega(\bar{q}_1) &= 1 + \pi_{P1}(q_{N1}) - \pi_{P1}(q_{P1}) + \pi_{N1}(\bar{q}_1) - \pi_{N1}(q_{N1}) \\ &= 1 + \frac{Y_1}{\bar{p}_1^{1-\sigma}}(\mu-1)\mu^{-\sigma}\left[q_{N1}^{\sigma-1} - q_{P1}^{\sigma-1}\right] + \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right](\mu-1)\mu^{-\sigma}\left[\bar{q}_1^{\sigma-1} - q_{N1}^{\sigma-1}\right]\end{aligned}$$

$$\begin{aligned}\pi_{P1}(q) &= \frac{Y_1}{\bar{p}_1^{1-\sigma}}(\mu-1)\mu^{-\sigma}\left(\frac{w_1}{q_k}\right)^{1-\sigma} \\ \pi_{N1}(q) &= \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right](\mu-1)\mu^{-\sigma}\left(\frac{w_1}{q_k}\right)^{1-\sigma} \\ \pi_{N2}(q) &= \left[\frac{Y_1}{\bar{p}_1^{1-\sigma}} + \frac{Y_2}{\bar{p}_2^{1-\sigma}}\right](\mu-1)\mu^{-\sigma}\left(\frac{w_2}{q_k}\right)^{1-\sigma} \\ \pi_{P2}(q) &= \frac{Y_2}{\bar{p}_2^{1-\sigma}}(\mu-1)\mu^{-\sigma}\left(\frac{w_2}{q_k}\right)^{1-\sigma}\end{aligned}$$

10. As $N_1 + P_1$ remains unchanged, there is no change in $\bar{q}_{P1}$. Thus the only effect comes through a decrease in q_{N1}. As the highest wage depends negatively on q_{N1}, this leads to an increase in inequality. What happens is that the returns to managerial quality increase over the range $[q_{N_1}, q_{N_1+dN_1}]$.

11. You could go through a nightmare of algebra if you wanted to.