14.462 Advanced Macroeconomics
Spring 2004

**Problem Set 4 Solution**

1. It is easy to check that $p_t = \varepsilon_t$ is the (bubbleless) REE equilibrium. Next consider the learning process. Substituting the model equation into the learning equation yields

$$p^e_{t+1} = p^e_t + \frac{1}{g_t}\left(\alpha p^e_t + \varepsilon_{t-1} - p^e_t\right) = h_t p^e_t + \frac{\varepsilon_{t-1}}{g_t}.$$

By induction one obtain

$$p^e_{t+1} = p^e_0 \prod_{s=0}^{t} h_s + \sum_{s=1}^{t+1} \frac{\varepsilon_{s-2}}{g_{s-1}} \prod_{\tau=s}^{t} h_\tau$$

Using this formula and the fact that the $\varepsilon_t$ are iid, we can compute

$$\mathbb{E}\left[p^e_{t+1} - \mathbb{E}_{t,REE} p_{t+1}\right]^2 = (p^e_0)^2 \prod_{s=0}^{t} h^2_s + \sigma^2 \sum_{s=1}^{t+1} \frac{1}{g^2_{s-1}} \prod_{\tau=s}^{t} h^2_\tau = (p^e_0)^2 \prod_{s=0}^{t} h^2_s + \sigma^2 m_t$$

2. Necessity is obvious. To show sufficiency notice that $m_t \geq \prod_{\tau=1}^{t} h^2_\tau$, so $\lim_{t\to\infty} m_t = 0$ insures that the term in front of $p^e_0$ vanishes as $t \to \infty$.

3. We have

$$h^2_{t+1} m_t + \frac{1}{g^2_{t+1}}$$
$$= h^2_{t+1} \sum_{s=1}^{t+1} \frac{1}{g^2_{s-1}} \prod_{\tau=s}^{t} h^2_\tau + \frac{1}{g^2_{t+1}} = \sum_{s=1}^{t+1} \frac{1}{g^2_{s-1}} \prod_{\tau=s}^{t+1} h^2_\tau + \frac{1}{g^2_{t+1}} = \sum_{s=1}^{t+2} \frac{1}{g^2_{s-1}} \prod_{\tau=s}^{t+1} h^2_\tau = m_{t+1}$$

4. First notice that $m_t \geq \frac{1}{g^2_t}$ so $\lim_{t\to\infty} m_t = 0$ clearly requires that $\lim_{t\to\infty} g_t = +\infty$. Also $m_t \geq \frac{1}{g^2_{s-1}} \prod_{\tau=s}^{t} h^2_\tau$ for all $s \in \{1, \ldots, t+1\}$. It follows that $\lim_{t\to\infty} m_t = 0$ requires

$$\lim_{t\to\infty} \prod_{\tau=s}^{t} h_\tau = 0$$

for all $s \geq 1$, which is stronger than the requirement stated in the problem set since it must hold for all $s \geq 1$. Finally if $\alpha \geq 1$, then $h_t \geq 1$ for all $t$ which is inconsistent with $\lim_{t\to\infty} \prod_{\tau=s}^{t} h_\tau = 0$. Thus it must be the case that $\alpha < 1$.

5. Suppose

$$\lim_{t\to\infty}\prod_{\tau=s}^{t} h_\tau = 0$$

for all $s \geq 1$. The goal is to show that $\sum_0^{+\infty} \frac{1}{g_t} = +\infty$.

6. Pick $T$ such that $t \geq T$ implies $\frac{1-\alpha}{g_t} < \frac{1}{2}$ and define $a_n = \frac{1-\alpha}{g_{T+n}}$ for $n \geq 0$. We want to show that $\prod_{n=0}^{\infty}(1-a_n) = 0$ if and only if $\sum_{n=0}^{\infty} a_n = +\infty$. This follows from some results on convergence of infinite products, see the book *Mathematical Analysis, 2ed* by Apostol, pp. 206–209. One direction is not so difficult: as $1 - a_n \leq e^{-a_n}$, we have $\prod_{n=0}^{\infty}(1-a_n) \leq e^{-\sum_{n=0}^{\infty} a_n}$. At least to me it seems that the other direction is a bit more involved, but perhaps you found an easy proof. Otherwise take a look at the book by Apostol.

7. If $g_t$ is a constant, then $\lim_{t\to\infty} g_t = +\infty$ fails. If $g_t = \frac{1}{(t+1)^2}$, then $\sum_0^{+\infty} \frac{1}{g_t} = +\infty$ fails.

8.