

Problem Set 5 Solution

Problem 1

1. The best response function is

$$g(K, \theta) = \begin{cases} 1 & \theta \geq \log(c) - \log(1 + K^\gamma) \\ 0 & \theta < \log(c) - \log(1 + K^\gamma) \end{cases}$$

We have $\underline{\theta} = \log\left(\frac{c}{2}\right)$ and $\bar{\theta} = \log(c)$. If $\theta < \underline{\theta}$, then it is best not to invest even if everybody else does. If $\theta \geq \bar{\theta}$, then it is optimal to invest even if nobody else does.

In a monotone equilibrium for each value of z there is a threshold $x^*(z)$ such that an agent invests if and only if $x \geq x^*(z)$. For a given value of θ and z , the fraction of agents investing is then given by

$$K(\theta, z) = \Phi(\sqrt{\alpha_x}(\theta - x^*(z))).$$

Let

$$H(x^*, x, z) = \mathbb{E} [e^\theta (1 + \Phi(\sqrt{\alpha_x}(\theta - x^*)))^\gamma - c|x, z]$$

be the expected utility from investing conditional on having observed signals x and z if the threshold of other agents for investing is x^* . To be an equilibrium, $x^*(z)$ must satisfy $H(x^*(z), x^*(z), z) = 0$. Since

$$\theta|x, z \sim \mathcal{N}(\delta x + (1 - \delta)z, \alpha^{-1})$$

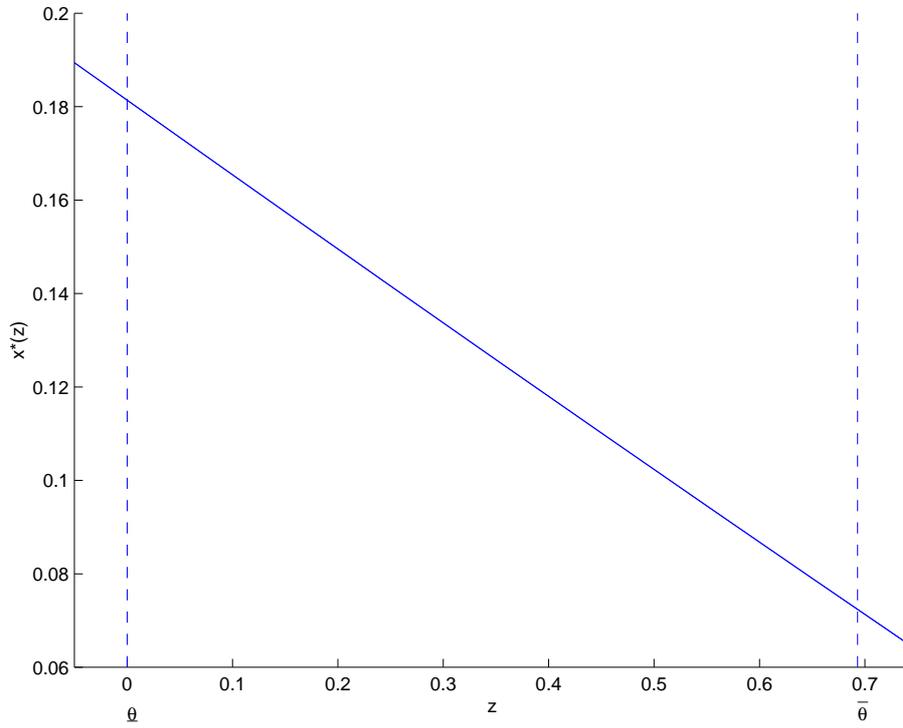
where $\delta = \frac{\alpha_x}{\alpha_x + \alpha_z}$ and $\alpha = \alpha_x + \alpha_z$ we can write this condition as

$$\int_{-\infty}^{\infty} e^\theta (1 + \Phi(\sqrt{\alpha_x}(\theta - x^*(z))))^\gamma \sqrt{\frac{\alpha}{2\pi}} e^{-\frac{1}{2}\alpha(\theta - \delta x^*(z) - (1-\delta)z)^2} d\theta = c$$

This condition can be used to solve numerically for equilibrium values of $x^*(z)$. With $\alpha_x = 10$ and $\alpha_z = 1$ the equilibrium is unique. The function $x^*(z)$ is plotted in Figure 1.

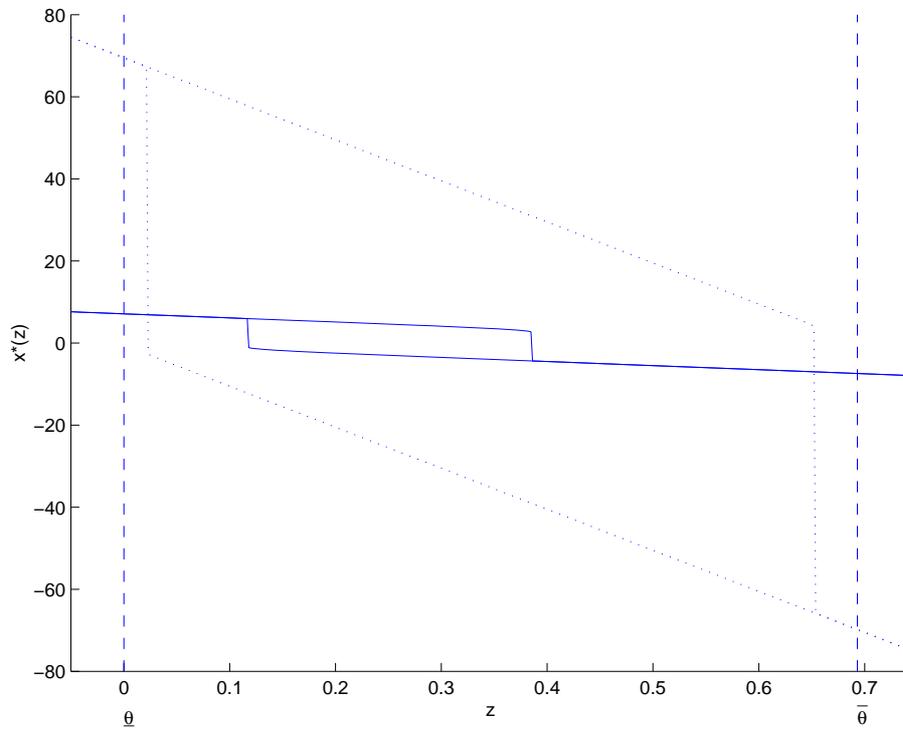
In the case $\alpha_x = 1$ and $\alpha_z = 10$ we have multiplicity for a range of values (\underline{z}, \bar{z}) . The solid line in Figure 2 plots the correspondence $x^*(z)$ for this case.

Figure 1: $x^*(z)$ for $\alpha_x = 10$ and $\alpha_z = 1$



2.

Figure 2: $x^*(z)$ for $\alpha_x = 1$ and $\alpha_z = 10$ (and $\alpha_z = 100$)



3.

4. The dotted line in Figure 2 is for the case $\alpha_x = 1$ and $\alpha_z = 100$. One can see that the multiplicity region (\underline{z}, \bar{z}) approaches $(\underline{\theta}, \bar{\theta})$ as α_z increases.

Problem 2

1. In the first stage agents will attack if $x \leq x_1^*$. Thus

$$A_1(\theta) = \Phi[\sqrt{\alpha_x}(x_1^* - \theta)].$$

Thus the public signal takes the form

$$z = \Phi^{-1}(A_1(\theta)) + v = \sqrt{\alpha_x}(x_1^* - \theta) + v.$$

It is convenient to consider the linear transformation

$$\tilde{z} = -\frac{1}{\sqrt{\alpha_x}}z + x_1^* = \theta - \frac{v}{\sqrt{\alpha_x}} = \theta + \tilde{v}$$

where $\tilde{v} = -\frac{v}{\sqrt{\alpha_x}}$ is distributed $\mathcal{N}(0, \frac{1}{\alpha_z \alpha_x})$. Thus the signal \tilde{z} has precision $\alpha_{\tilde{z}} = \alpha_z \alpha_x$.

2. Now we're looking for monotone equilibria of the second stage. This problem is almost the problem considered in class with one complication: agents that have decided to attack in the first stage cannot reverse their decision. There are now two situations that can occur in the second stage. Either some agents join the attack, in which case all agents with private signal below some threshold $x_I^*(\tilde{z})$ participate in the attack and the agent with signal $x_I^*(\tilde{z})$ is indifferent. Or nobody joins the attack, so everybody below x_1^* participates in the attack but an agent with signal x_1^* is not indifferent but would rather not attack but cannot reverse his decision. In either case there is a threshold $x_p^*(\tilde{z})$ below which agents participate in the attack. Then the overall size of the attack is given by

$$A(\theta, \tilde{z}) = \Phi(\sqrt{\alpha_x}(x_p^*(z) - \theta)).$$

Regime change occurs if and only if

$$\begin{aligned} & \theta \leq \Phi(\sqrt{\alpha_x}(x_p^*(z) - \theta)) \\ \iff & \Phi^{-1}(\theta) \leq \sqrt{\alpha_x}(x_p^*(z) - \theta) \\ \iff & \theta + \frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta) \leq x_p^*(z) \\ \iff & X(\theta) \leq x_p^*(z) \end{aligned}$$

where

$$X(\theta) \equiv \theta + \frac{1}{\sqrt{\alpha_x}} \Phi^{-1}(\theta).$$

The function X is strictly increasing. Let $\theta_p^*(\tilde{z})$ be the level of fundamentals at which the inequality above holds with equality, that is $X(\theta_p^*(\tilde{z})) = x_p^*(\tilde{z})$. Then regime change occurs if and only if $\theta \leq \theta_p^*(\tilde{z})$. Let's write $\Theta = X^{-1}$, then we can write $\theta_p^*(\tilde{z}) = \Theta(x_p^*(\tilde{z}))$. That is, for a given participation threshold $x_p^*(\tilde{z})$, the function Θ will give us the threshold of fundamentals below which regime change occurs.

Now let $H(x_p^*(\tilde{z}), x, \tilde{z})$ be the expected utility from attacking of an agent conditional on having observed a private signal x and a public signal \tilde{z} if all other agents below the participation threshold $x_p^*(\tilde{z})$ participate in the attack. So

$$H(x_p^*(\tilde{z}), x, \tilde{z}) \equiv bP[\theta \leq \Theta(x_p^*(\tilde{z})) | x, \tilde{z}] - c.$$

In the standard Morris-Shin model the set of equilibria is given by the solutions to the equation

$$H(x, x, \tilde{z}) = 0$$

Here things are a bit more involved due to irreversibility. First suppose there is a unique solution to this equation. Here the condition that insures uniqueness for all values of \tilde{z} is $\frac{\alpha_{\tilde{z}}}{\sqrt{\alpha_x}} \leq \sqrt{2\pi}$, which can be written as $\alpha_z \sqrt{\alpha_x} \leq \sqrt{2\pi}$. Notice that a more precise private signal now also helps to generate multiplicity, because this means that the public signal is more precise. If this condition fails there is a range of values $[\underline{z}, \bar{z}]$ for \tilde{z} over which there is multiplicity. For now we have assumed that we're outside of this multiplicity range. Denote the unique solution as $x_U^*(\tilde{z})$. If $x_U^*(\tilde{z}) \geq x_1^*$ we have found an equilibrium. It will be useful to describe the equilibrium by two thresholds, which turn out to be the the same in this case. First there is the threshold below which agents participate in the attack, denoted as $x_p^*(\tilde{z})$ which is equal to $x_U^*(\tilde{z})$ in this case. Second there is the threshold at which an agent is indifferent, denoted as $x_I^*(\tilde{z})$ and also equal to $x_U^*(\tilde{z})$ in this case. The two thresholds are no longer the same if $x_U^*(\tilde{z}) < x_1^*$. Now the participation threshold is $x_p^*(\tilde{z}) = x_1^*$: nobody joins the attack but those with signal below x_1^* are stuck with their earlier decision. Notice that due to uniqueness and the fact that $x_U^*(\tilde{z}) < x_1^*$ we must have $H(x_1^*, x_1^*, \tilde{z}) < 0$, so an agent with private signal x_1^* would like to reverse his decision but cannot due so. Thus the indifference threshold which is determined by the condition $H(x_1^*, x_I^*(\tilde{z}), \tilde{z}) = 0$ must satisfy $x_I^*(\tilde{z}) < x_1^*$.¹ Why is it important to determine the indifference threshold? After all, in equilibrium everybody below x_1^* has already decided to attack. The answer is that when deciding on whether to attack in the first stage, agents must

¹One can easily solve for this number explicitly, the condition is

$$bP[\theta \leq \Theta(x_1^*) | x_I^*(\tilde{z}), \tilde{z}] - c = 0.$$

contemplate in what situations they would like to attack in the second stage given that everybody else below x_1^* attacks in the first stage.

Now suppose we are in the case of multiplicity. I will focus on the typical case with three solutions to the equation $H(x, x, \tilde{z}) = 0$, the extension to the borderline cases with two equilibria is straightforward. Denote the three solutions as $x_L^*(\tilde{z}) < x_M^*(\tilde{z}) < x_H^*(\tilde{z})$. We have several cases to consider:

- (a) $x_1^* \leq x_L^*(\tilde{z})$: all three values $x_L^*(\tilde{z})$, $x_M^*(\tilde{z})$ and $x_H^*(\tilde{z})$ are equilibria (with identical participation and indifference threshold). Of course as usual the intermediate equilibrium is unstable and thus of little interest.
 - (b) $x_L^*(\tilde{z}) < x_1^* < x_M^*(\tilde{z})$: now the value $x_L^*(\tilde{z})$ no longer constitutes an equilibrium. Since x_1^* is larger than $x_L^*(\tilde{z})$ but still below $x_M^*(\tilde{z})$ we have $H(x_1^*, x_1^*, \tilde{z}) < 0$, so we get an equilibrium with participation threshold $x_P^*(\tilde{z}) = x_1^*$ and indifference threshold $x_I^*(\tilde{z}) < x_1^*$ solving $H(x_1^*, x_I^*(\tilde{z}), \tilde{z}) = 0$. The other two values $x_M^*(\tilde{z})$ and $x_H^*(\tilde{z})$ still give equilibria with identical participation and indifference threshold.
 - (c) $x_M^*(\tilde{z}) < x_1^* < x_H^*(\tilde{z})$: now the intermediate value no longer gives an equilibrium. Moreover, since x_1^* is now between the intermediate and the large value, we have $H(x_1^*, x_1^*, \tilde{z}) > 0$. This means that given a participation threshold x_1^* , agents with signal x_1^* are happy to attack in the second stage. So there is no equilibrium with participation threshold x_1^* and indifference threshold $x_I^*(\tilde{z}) < x_1^*$. We are only left with the equilibrium $x_H^*(\tilde{z})$ with identical participation and indifference threshold.
 - (d) $x_H^*(\tilde{z}) < x_1^*$: now even the large value no longer gives an equilibrium. Once again we have $H(x_1^*, x_1^*, \tilde{z}) < 0$, so we have an equilibrium with participation threshold $x_P^*(\tilde{z}) = x_1^*$ and indifference threshold $x_I^*(\tilde{z}) < x_1^*$
3. Suppose from solving the second stage we have the equilibrium participation threshold $x_P^*(\tilde{z}, x_1^*)$ and indifference threshold $x_I^*(\tilde{z}, x_1^*)$. Here I have made the dependence of the second stage equilibrium on x_1^* explicit. Of course in the case of multiplicity we can choose among the different second stage equilibria, which will translate into multiplicity in the overall game.

First let's figure out the posterior distribution that an agent has over the public signal \tilde{z} after observing his private signal x . As $\theta = x - \xi$ we have $\tilde{z} = \theta + \tilde{v} = x - \xi + \tilde{v}$ we

As usual let $\alpha = \alpha_x + \alpha_z = \alpha_x(1 + \alpha_z)$ and $\delta = \frac{\alpha_x}{\alpha_x + \alpha_z} = \frac{1}{1 + \alpha_z}$. Then we get the condition

$$b [1 - \Phi(\sqrt{\alpha}(\delta x + (1 - \delta)\tilde{z} - \Theta(x_1^*)))] = c \iff x_I^*(\tilde{z}) = \frac{1}{\delta} \left[\frac{1}{\sqrt{\alpha}} \Phi^{-1} \left(1 - \frac{c}{b} \right) - (1 - \delta)\tilde{z} + \Theta(x_1^*) \right]$$

have

$$\begin{aligned}\mathbb{E}[\tilde{z}|x] &= x - \mathbb{E}[\xi|x] + \mathbb{E}[\tilde{v}|x] = x \\ \text{Var}[\tilde{z}|x] &= \text{Var}[\xi|x] + \text{Var}[\tilde{v}|x] = \frac{1}{\alpha_x} + \frac{1}{\alpha_x \alpha_z} = \frac{1 + \alpha_z}{\alpha_x \alpha_z}.\end{aligned}$$

$$\tilde{z}|x \sim \mathcal{N}\left(x, \frac{1 + \alpha_z}{\alpha_x \alpha_z}\right),$$

so the density is

$$f(\tilde{z}|x) = \sqrt{\frac{\alpha_x \alpha_z}{(1 + \alpha_z) 2\pi}} e^{-\frac{1}{2} \frac{\alpha_x \alpha_z}{1 + \alpha_z} (\tilde{z} - x)^2}$$

As usual we have

$$P[\theta \leq \Theta(x_P^*(\tilde{z}, x_1^*))|x, \tilde{z}] = 1 - \Phi(\sqrt{\alpha}(\delta x + (1 - \delta)\tilde{z} - \Theta(x_P^*(\tilde{z}, x_1^*))))$$

Investing in the first stage yields expected utility

$$\int (1 + \beta)[bP[\theta \leq \Theta(x_P^*(\tilde{z}, x_1^*))|x, \tilde{z}] - c]f(\tilde{z}|x)d\tilde{z}$$

while waiting gives expected utility

$$\int \beta[bP[\theta \leq \Theta(x_P^*(\tilde{z}, x_1^*))|x, \tilde{z}] - c]\mathcal{I}[x \leq x_I^*(\tilde{z}, x_1^*)]f(\tilde{z}|x)d\tilde{z}.$$

Notice that to compute the value of waiting we need to know the indifference threshold $x_I^*(\tilde{z}, x_1^*)$: in the first agent can choose a value different from x_1^* . In order to compute the utility of choosing a value below x_1^* we need to know when the agent would be willing to invest in the second stage.

The equilibrium thresholds x_1^* must satisfy the condition

$$\int [(1 + \beta) - \beta\mathcal{I}[x_1^* \leq x_I^*(\tilde{z}, x_1^*)]] [bP[\theta \leq \theta_p^*(\tilde{z}, x_1^*)|x_1^*, \tilde{z}] - c]f(\tilde{z}|x_1^*)d\tilde{z} = 0$$

We have determined all the pieces in this equation and can now compute equilibria numerically. I haven't attempted to prove whether the solution for x_1^* is unique for a given selection of second stage equilibria, but in my numerical examples it is.

First I compute some equilibria for the parameter values $\alpha_x = 1$, $\alpha_z = 3$ (so the condition for multiplicity is satisfied), $\beta = 0.8$, $b = 1$ and $c = 0.5$. I compute two equilibria. In the first I select the lowest second stage equilibrium for all values of \tilde{z} , in the second I always select the highest second stage equilibrium. Figures 3 and 4 show the functions $x_I^*(\tilde{z})$ and $x_P^*(\tilde{z})$ for the first equilibrium. Let's first discuss figure 3. The two vertical dashed lines are \underline{z} and \bar{z} , the values of the public signal \tilde{z} for

Figure 3: Low second stage equilibrium, function $x_I^*(\tilde{z})$

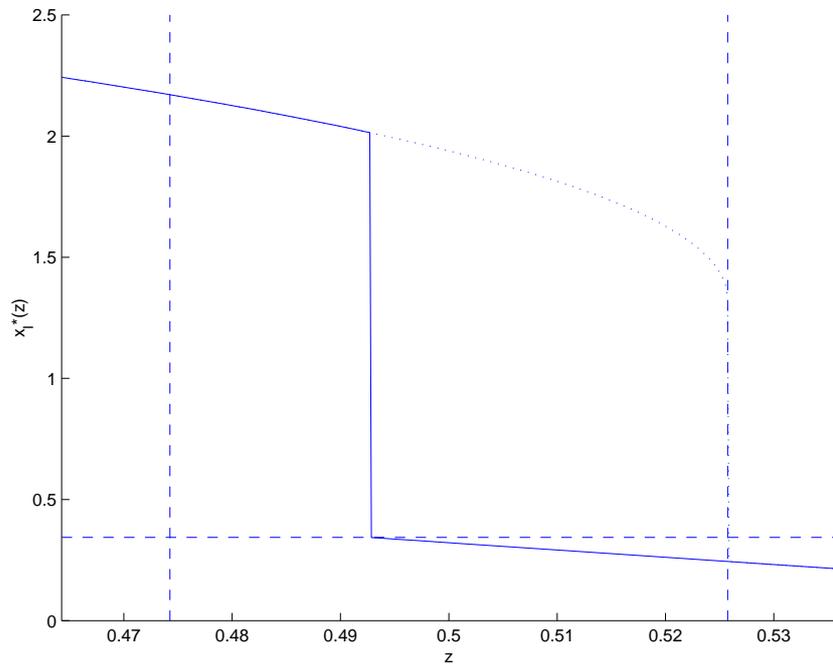
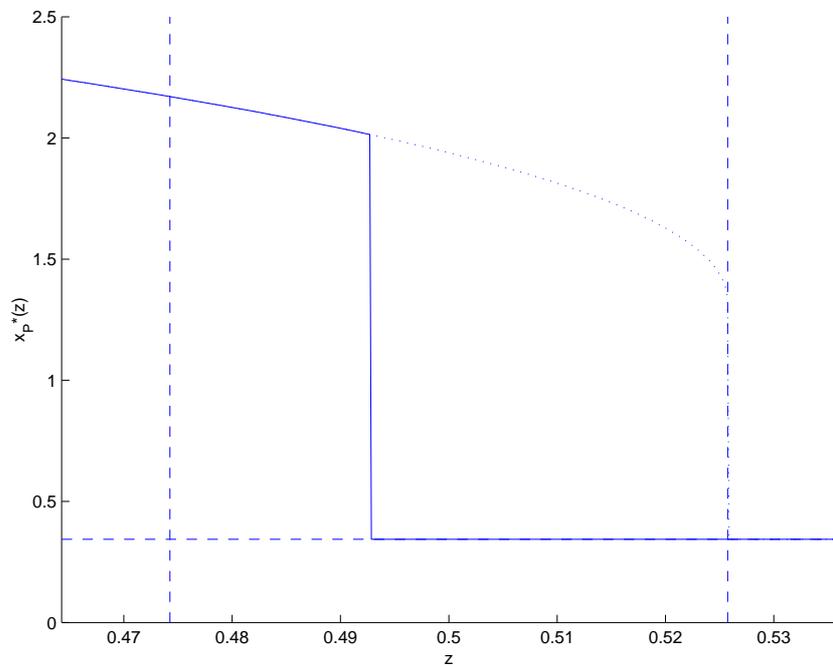


Figure 4: Low second stage equilibrium, function $x_P^*(\tilde{z})$



which we would have multiplicity in the standard Morris-Shin model. The horizontal dashed line is the equilibrium first stage threshold x_1^* . The solid discontinuous line is $x_I^*(\tilde{z})$. The dotted continuation of the left part of $x_I^*(\tilde{z})$ indicates the not selected high second stage equilibrium. Notice that we do not have multiplicity over the entire range $[\underline{z}, \bar{z}]$. Over that range we are in case (c): the lowest solution of $H(x, x, \tilde{z}) = 0$ is below x_1^* ; at the same time $H(x_1^*, x_1^*, \tilde{z}) > 0$, so with a participation threshold x_1^* more agents would like to joint the attack; as a consequence, only the high equilibrium survives. As we increase \tilde{z} , $H(x_1^*, x_1^*, \tilde{z})$ falls and the discontinuity occurs at the point when $H(x_1^*, x_1^*, \tilde{z}) = 0$. From now on we have a low equilibrium with participation threshold x_1^* (this is shown in figure 4) and an indifference threshold $x_I^*(\tilde{z})$ (which is shown in figure 3).

Figures 5 and 6 shows the same graphs when the high second stage equilibrium is selected. Selecting the high second stage equilibrium is associated with a slightly higher (imperceptible in the figures) value of the first stage threshold x_1^* . Figures 7 and 8 show the effect on the functions $x_I^*(\tilde{z})$ and $x_P^*(\tilde{z})$ of a reduction in β to 0.5. I consider the more interesting case in which the low second stage equilibrium is selected. A lower discount rate makes it more attractive to attack in the first stage, so x_1^* increases (the old value is given by the dashed horizontal line, the new one by the dash-dotted line). Figure 7 shows the effect on the $x_I^*(\tilde{z})$ schedule. The old schedule is solid, the new one dotted. The increase in β has of course no effect on the location of the high solution to $H(x, x, \tilde{z}) = 0$. But the point of discontinuity shifts to the right and the indifference threshold shifts up: since now in the low second stage equilibrium more people are stuck with having attacked (as shown in figure 8), people would be more willing to join the attack in the second stage.

Figure 5: High second stage equilibrium, function $x_I^*(\tilde{z})$

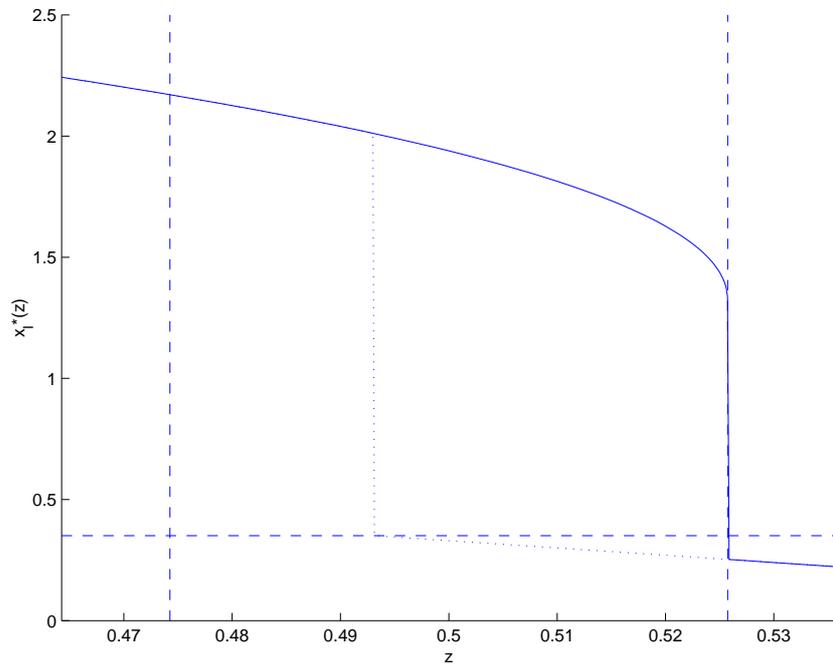


Figure 6: High second stage equilibrium, function $x_P^*(\tilde{z})$

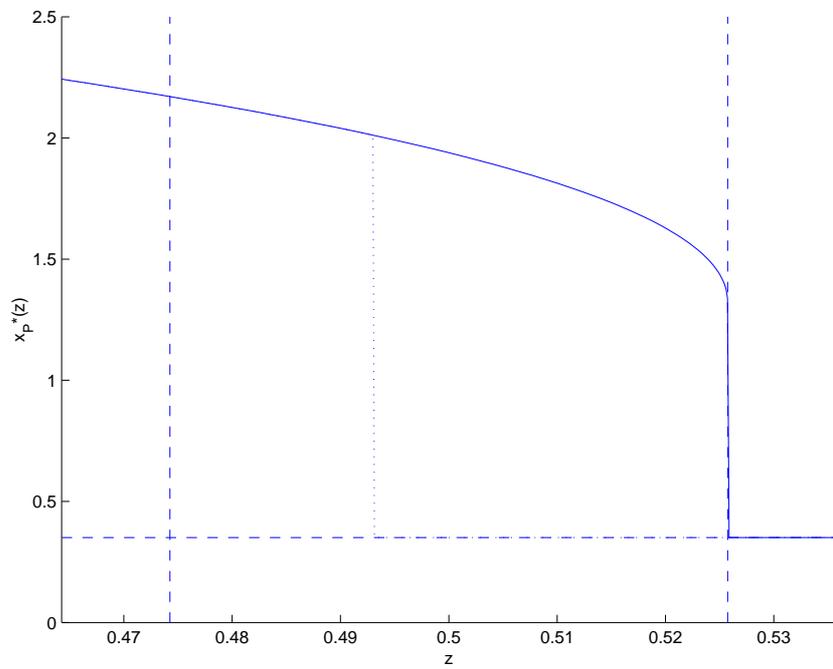


Figure 7: Reduction in β , function $x_I^*(\tilde{z})$

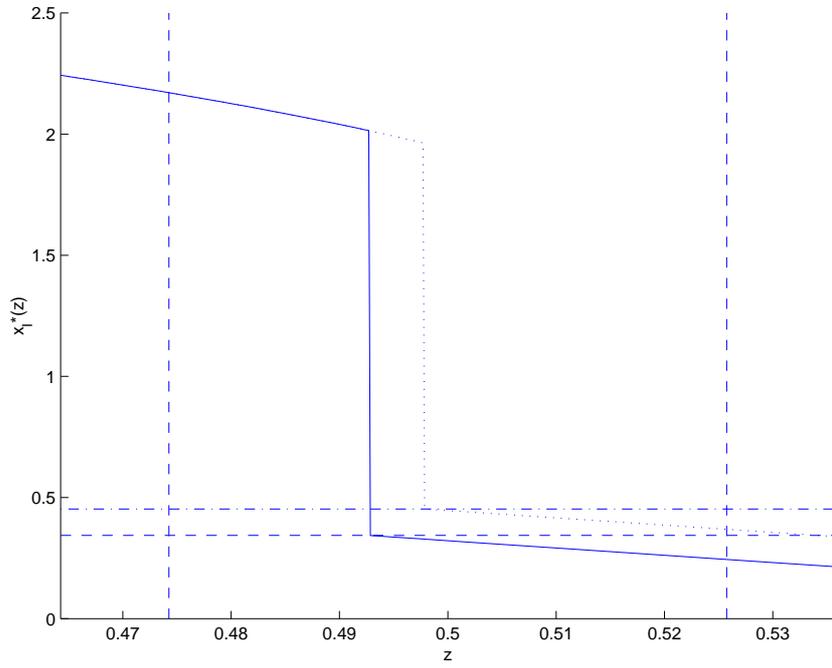


Figure 8: Reduction in β , function $x_P^*(\tilde{z})$

