14.462 Advanced Macroeconomics
Spring 2004

Problem Set 6
(due May 7)

Problem 1
The economy is populated by a continuum of measure one of agents, indexed by i and uniformly distributed over the $[0,1]$ interval. Agents are risk neutral with utility

$$u_i = Ak_i - \frac{1}{2}k_i^2$$

where k_i is the individual investment of agent i, A is the return to investment, and $\frac{k_i^2}{2}$ is the cost of investment. Let

$$K = \int_0^1 k_i di$$

denote the aggregate level of investment. The return to investment is given by

$$A = (1-\alpha)\theta + \alpha K$$

where $\alpha \in \left[0, \frac{1}{2}\right)$. The random variable θ parametrizes the fundamentals of the economy. If $\alpha > 0$ there is a complementarity in that the return to individual investment is increasing in the aggregate level of investment, and the parameter α captures the degree of complementarity.
The fundamentals θ are not known at the time investment decisions are made. Furthermore, agents have heterogeneous beliefs about θ. The common prior is uniform over $\mathbb{R}$. Agent i has private information

$$x_i = \theta + \sigma_x \xi_i$$

and there is public information

$$y = K + \sigma_y u.$$

The random variables ξ_i, $i \in [0,1]$ and u are standard normal and independent as well as independent of θ. The precisions of the two sources of information are denoted as $\pi_x = \sigma_x^{-2}$, and $\pi_y = \sigma_y^{-2}$, respectively. Let social welfare be given by a utilitarian aggregator

$$w = \int_0^1 u_i di.$$

1. Check that it is an equilibrium for investment to takes the form

$$k_i = \beta x_i + (1-\beta)y.$$

Determine the coefficient β and describe how it varies with the degree of complementarity α and the two precisions π_x and π_y. Provide an intuitive explanation of your findings.

2. How do heterogeneity $\text{Var}(k_i|\theta, y)$ and volatility $\text{Var}(K|\theta)$ vary with the three parameters α, π_x and π_y. Provide intuition.

3. Show that social welfare conditional on fundamentals $\mathbb{E}[w|\theta]$ is a linear function of heterogeneity and volatility. Use your previous results to discuss how social welfare varies with the parameters and provide intuition for your results.

4. Now suppose there is a second source of public information

$$z = \theta + \sigma_z \varepsilon$$

where ε is standard normal and independent of θ, u and the ξ_i, $i \in [0,1]$. How are the answers to parts 1.-3. affected?

Problem 2 (A simple Model of Savings)
Consider the problem of a consumer who wants to maximize the following program:

$$\max_{\{c_t\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \tag{1}$$

$$\begin{aligned} \text{s.t. (i)}\ w_t &= e_t + R_t b_t \\ w_t &= c_t + b_{t+1} \qquad (2) \\ \text{(ii)}\ w_o &> 0 \end{aligned}$$

The endowment shock e_t and the interest rate R_t are i.i.d. Don't worry about the non-negativity constraint on consumption.

1. Rewrite the problem in recursive form.

2. Without solving for the value function, derive the first order condition (FOC) and the envelope condition (EC). Combine the two to obtain the Euler Equation (EE).

3. Assume in this section that the endowment shock is 0 in all periods. Make a guess for a value function. Using this guess, derive the consumption function. Using the EE, solve for the constant. Then replace back into the Bellman equation, and verify that you indeed found the value function.

4. Assume now that the endowment shock is stochastic, and that the interest rate is deterministic and $R_t = R$. Use the EE to analyze consumption growth. What happens if $R\beta = 1$? What is the R that makes expected consumption growth zero? Discuss the implications of uninsurable risk.

5. Assume that R_t is stochastic, and that the consumer is the representative agent of the economy. Assume that endowment is stochastic, and that the asset is in zero net supply. Use the EE to price the asset when there is only aggregate risk. Discuss (but do not solve) the case with only idiosyncratic risk.

6. Now assume that there is no uncertainty, that the interest rate is constant, and the endowment shock 0. Solve for the value function, and the optimal consumption and wealth path as a function of initial wealth. Using the optimal consumption path that you derived from the recursive approach, derive the value function by replacing consumption in expected utility. What condition in γ do you need to make sure that the solution is indeed optimal? Discuss.