14.472 Handout on varying life expectancy

From Diamond Munich Lectures Chapter 7. Models of Retirement Incentives with Varying Life Expectancy

We use a four-period model where everyone works in the first period, no one works in the third or fourth periods and some people do work and some do not in period two. There is no savings and a fully-rational forward-looking retirement decision. Everyone survives through period three, with some retirees dying between periods three and four. Everyone is assumed to know their survival probabilities. It is assumed that all workers have the same earnings potential.

Moral hazard constraint

Two different utility functions when working and not: $u[x] - a$, and $v[c]$, functions of consumption when working, x, and consumption when retired, c. We assume both utility functions are concave.

Assume that the moral hazard condition is met: that if marginal utilities are equated, utility would be higher without work for all values of a*:*

$$\text{If } u'[x] = v'[c], \text{then } u[x] < v[c]. \quad (1)$$

Note that this is a "normality" assumption. If a worker were just indifferent to work, $u[x] = v[c]$, then small equal increases in consumption whether working or not would raise utility in the two states by $u'[x]$ and $v'[c]$. Since $u'[x] < v'[c]$, this would give lower utility if working and so result in retirement.

Assume that at the optimum there is an interior solution - some, but not complete, second-period work. Assume that real social security benefits vary with age at retirement but not age when getting each benefit.

Let p be the probability of worker-p surviving to period 4. Let a_p $(a_p \geq 0)$ be the additive portion of the disutility of labor in periods 1 and 2 for someone who has the probability p of surviving to period 4. We assume that a_p is nonincreasing in p, i. e., $da_p/dp \leq 0$. That is, those with longer lives are more capable of working in period 2, as measured by disutility. Assume p can vary between p_0 and p_1, and has distribution $F[p]$ and density $f[p]$.

We assume additive preferences and zero utility discount and interest rates. Subscripts on consumption when working refer to the period, while those on consumption when retired refer to the period of retirement. A worker chooses between two plans. With no work in period 2, expected lifetime utility for worker-p is $u[x_1] - a_p + (2+p)\, v[c_1]$. With work in period 2, expected lifetime utility for worker-p is $u[x_1] + u[x_2] - 2a_p + (1+p)\, v[c_2]$.

By assumption there is an interior solution - some people retire early and some retire late. Thus there is a marginal worker with survival probability p^* and disutility that equates lifetime utility with the two plans, and so satisfies

$$a_{p^*} = u[x_2] + (1+p^*)v[c_2] - (2+p^*)v[c_1]. \tag{2}$$

Differentiating implicitly, we have:

$$\begin{aligned} \frac{dp^*}{dx_2} &= -u'[x_2]/D, \\ \frac{dp^*}{dc_1} &= (2+p^*)v'[c_1]/D, \\ \frac{dp^*}{dc_2} &= -(1+p^*)v'[c_2]/D, \end{aligned} \tag{3}$$

where $D = v[c_2] - v[c_1] - da_p/dp$. There is a unique solution provided $c_2 \geq c_1$. Moreover, with this condition, we have $D > 0$. The optimum has this property.

The implicit tax on work measures the impact on the government budget of additional work. The implicit tax on second-period work for worker-p is the marginal product plus expected future benefits if not working less the sum of current consumption and expected future benefits if working. Thus the implicit tax for worker-p is:

$$T[p] = n - x_2 - (1+p)c_2 + (2+p)c_1. \tag{4}$$

With $c_2 > c_1$, implicit taxes are less for those with longer life expectancies and larger for those with shorter life expectancies.

Social welfare maximization, assuming simple addition of lifetime utilities, is now:

$$\begin{aligned} \text{Maximize}_{x,c} \quad & \int_{p_0}^{p^*} \{u[x_1] - a_p + (2+p)v[c_1]\} dF[p] \\ & + \int_{p^*}^{p_1} \{u[x_1] + u[x_2] - 2a_p + (1+p)v[c_2]\} dF[p] \\ \text{subject to:} \quad & E + \int_{p_0}^{p^*} \{x_1 - n + (2+p)c_1\} dF[p] \\ & + \int_{p^*}^{p_1} \{x_1 + x_2 - 2n + (1+p)c_2\} dF[p] \leq 0 \\ \text{where } p^* \text{ satisfies} \quad & a_{p^*} = u[x_2] + (1+p^*)v[c_2] - (2+p^*)v[c_1] \end{aligned} \tag{5}$$

Deriving the FOC and rearranging terms, as shown in the proof, the FOC can be written as:

$$\begin{aligned}
u'[x_1] &= \lambda, \\
u'[x_2] &= \lambda/\{1+\lambda\frac{T[p^*]}{D}\frac{f[p^*]}{1-F[p^*]}\}, \\
v'[c_1] &= \lambda/\{1-\lambda\frac{T[p^*]}{D}\frac{f[p^*]}{F[p^*]}\frac{2+p^*}{2+P_0[p^*]}\}, \\
v'[c_2] &= \lambda/\{1+\lambda\frac{T[p^*]}{D}\frac{f[p^*]}{1-F[p^*]}\frac{1+p^*}{1+P_1[p^*]}\}.
\end{aligned} \tag{6}$$

where $P_0[p]$ and $P_1[p]$ are the average survival probabilities for those below and above p:

$$\begin{aligned}
P_0[p] &= \int_{p_0}^{p} sdF[s]/F[p], \\
P_1[p] &= \int_{p}^{p_1} sdF[s]/(1-F[p]).
\end{aligned} \tag{7}$$

Thus we have:

$$P_0[p] < p < P_1[p]. \tag{8}$$

Naturally the FOC distinguish the average life expectancies of the early and late retirees.

The Lagrangian equals the marginal utility of consumption in period one since there are no incentive effects associated with this net wage. Each of the other three consumption levels have an effect on retirement decisions, and so on the resource constraint. Increasing the net wage in period two or the level of benefits for late retirees encourages more work and so improves the resource constraint given a positive implicit tax on work. Increasing the benefit for early retirees has the opposite effect. Naturally these incentives affect the optimal levels of consumption. The qualitative results are summarized in:

Theorem. At the optimum we have: $T[p^*] > 0$, $D > 0$,

$$u'[x_2] < v'[c_2] < u'[x_1] < v'[c_1] \tag{9}$$

implying

$$c_2 > c_1,\ x_2 > x_1. \tag{10}$$

Corollary. If u and v differ by an additive constant, then

$$x_2 > c_2 > x_1 > c_1. \tag{11}$$

Contrasting this optimum with the case with uniform life expectancies, we see that some workers are taxed and some may be subsidized at the optimum. The levels of consumption provided depend on the life expectancies of the different groups receiving different benefits. The relationship among probabilities, $P_0[p] < p < P_1[p]$, gives us the size of the terms in the FOC that would equal 1 if life expectancies were all the same: $\frac{2+p^*}{2+P_0[p^*]} > 1$ and $\frac{1+p^*}{1+P_1[p^*]} < 1$. Thus the variance in life expectancy would matter for the optimum, with mean held constant. Just accounting for these terms (and ignoring the impact of a change in the distribution of life expectancies on other terms in the FOC), more variance in life expectancies tends to lower both c_1 and c_2. The wage for working in period two is larger than the wage in period one and delaying retirement increases the retirement benefit in periods three and four.

Proof. The FOC as stated above come from solving the FOC:

$$\begin{aligned} u'[x_1] &= \lambda, \qquad (12) \\ (u'[x_2]-\lambda)(1-F[p^*]) &= \lambda T[p^*]f[p^*]\frac{dp^*}{dx_2} = -\lambda\frac{T[p^*]}{D}f[p^*]u'[x_2], \\ (v'[c_1]-\lambda)(2+P_0[p^*])F[p^*] &= \lambda T[p^*]f[p^*]\frac{dp^*}{dc_1} = \lambda\frac{T[p^*]}{D}f[p^*](2+p^*)v'[c_1], \\ (v'[c_2]-\lambda)(1+P_1[p^*])(1-F[p^*]) &= \lambda T[p^*]f[p^*]\frac{dp^*}{dc_2} = -\lambda\frac{T[p^*]}{D}f[p^*](1+p^*)v'[c_2], \end{aligned}$$

Lemma 1. If and only if $T[p^*]/D > 0$, then $u'[x_2] < v'[c_2]$, $v'[c_2] < u'[x_1] < v'[c_1]$.

Proof of Lemma 1.

$$\frac{u'[x_2]}{v'[c_2]} = \frac{K + (1+p^*)/(1+P_1[p^*])}{K+1} < 1, \tag{13}$$

where $K^{-1} = \lambda \frac{T[p^*]}{D} \frac{f[p^*]}{1-F[p^*]}$.

The first inequality follows from the sign of $K > 0$ and $(1+p^*)/(1+P_1[p^*]) < 1$. The rest follow from the FOC.

Lemma 2. If $T[p^*]/D > 0$, then $T[p^*] > 0$, $D > 0$.

Proof of Lemma 2. If $D < 0$, then $v[c_2] < v[c_1]$ since $da_p/dp < 0$.

From Lemma 1, $v'[c_2] < v'[c_1]$. This is a contradiction.

Proof of Theorem. If $T[p^*]/D > 0$, the results follow from the lemmas. If $T[p^*]/D \leq 0$, then we contradict the assumption that $a_{p^*} > 0$ by the following argument. If $T[p^*]$ is zero then all marginal utilities are equal and from the moral hazard constraint we have a contradiction. With $T[p^*] < 0$, Lemma 1 gives the same contradiction. A similar argument works with $T[p^*] > 0$ and $D < 0$. ■