14.581 International Trade
— Lecture 4: Assignment Models —

Today's Plan

1. Overview
2. Log-supermodularity
3. Comparative advantage based asignment models
4. Cross-sectional predictions
5. Comparative static predictions

1. Overview

Assignment Models in the Trade Literature

- Small but rapidly growing literature using assignment models in an international context:
  - Trade: Grossman Maggi (2000), Grossman (2004), Yeaple (2005), Ohnsorge Trefler (2007), Costinot (2009), Costinot Vogel (2010), Sampson (2012)
  - Offshoring: Kremer Maskin (2003), Antras Garicano Rossi-Hansberg (2006), Nocke Yeaple (2008), Costinot Vogel Wang (2011)
- **What do these models have in common?**
  - Factor allocation can be summarized by an assignment function
  - Large number of factors and/or goods
- **What is the main difference between these models?**
  - *Matching:* Two sides of each match in finite supply (as in Becker 1973)
  - *Sorting:* One side of each match in infinite supply (as in Roy 1951)

This Lecture

- I will restrict myself to sorting models, e.g. Ohnsorge and Trefler (2007), Costinot (2009), and Costinot and Vogel (2010)
- **Objectives:**
  1. Describe how these models relate to "standard" neoclassical models
  2. Introduce simple tools from the mathematics of complementarity
  3. Use tools to derive cross-sectional and comparative static predictions
- This is very much a methodological lecture. If you are interested in more specific applications, read the papers...

2. Log-Supermodularity

- **Definition 1** *A function* $g\colon X \to \mathbb{R}^+$ *is log-supermodular if for all* $x, x' \in X$, $g(\max(x, x')) \cdot g(\min(x, x')) \geq g(x) \cdot g(x')$
- **Bivariate example:**
  - If $g : X_1 \times X_2 \to \mathbb{R}^+$ is log-spm, then $x_1' \geq x_1''$ and $x_2' \geq x_2''$ imply

$$g(x_1', x_2') \cdot g(x_1'', x_2'') \geq g(x_1', x_2'') \cdot g(x_1'', x_2',).$$

  - If g is strictly positive, this can be rearranged as

$$g(x_1', x_2') / g(x_1'', x_2') \geq g(x_1', x_2'') / g(x_1'', x_2'') .$$

Log-supermodularity

Results

- **Lemma 1.** $g, h : X \to \mathbb{R}^+$ *log-spm* $\Rightarrow$ *gh log-spm*
- **Lemma 2.** $g : X \to \mathbb{R}^+$ *log-spm* $\Rightarrow$ $G(x_{-i}) = \int_{X_i} g(x)\, dx_i$ *log-spm*
- **Lemma 3.** $g : T \times X \to \mathbb{R}^+$ *log-spm* $\Rightarrow$
  $x^*(t) \equiv \arg\max_{x \in X} g(t, x)$ *increasing in t*

3. Sorting Models

Basic Environment

- Consider a world economy with:
  1. Multiple countries with characteristics $\gamma \in \Gamma$
  2. Multiple goods or sectors with characteristics $\sigma \in \Sigma$
  3. Multiple factors of production with characteristics $\omega \in \Omega$
- Factors are immobile across countries, perfectly mobile across sectors
- Goods are freely traded at world price $p(\sigma) > 0$

Technology

- Within each sector, factors of production are perfect substitutes

$$Q(\sigma,\gamma)=\int_{\Omega}A(\omega,\sigma,\gamma)L(\omega,\sigma,\gamma)d\omega,$$

- $A(\omega,\sigma,\gamma)\geq 0$ is productivity of ω-factor in σ-sector and γ-country
- **A1** $A(\omega,\sigma,\gamma)$ *is log-supermodular*
- A1 implies, in particular, that:

  1. High-γ countries have a comparative advantage in high-σ sectors
  2. High-ω factors have a comparative advantage in high-σ sectors

Factor Endowments

- $V(\omega, \gamma) \geq 0$ is inelastic supply of ω-factor in γ-country
- **A2** $V(\omega, \gamma)$ *is log-supermodular*
- A2 implies that:
  High-γ countries are relatively more abundant in high-ω factors
- Preferences will be described later on when we do comparative statics

4. Cross-Sectional Predictions

4.1 Competitive Equilibrium

- We take the price schedule $p(\sigma)$ as given [small open economy]
- In a competitive equilibrium, L and w must be such that:

1. Firms maximize profit

$$\begin{aligned} p(\sigma) A(\omega, \sigma, \gamma) - w(\omega, \gamma) &\leq 0, \text{ for all } \omega \in \Omega \\ p(\sigma) A(\omega, \sigma, \gamma) - w(\omega, \gamma) &= 0, \text{ for all } \omega \in \Omega \text{ s.t. } L(\omega, \sigma, \gamma) > 0 \end{aligned}$$

2. Factor markets clear

$$V(\omega, \gamma) = \int_{\sigma \in \Sigma} L(\omega, \sigma, \gamma) \, d\sigma, \text{ for all } \omega \in \Omega$$

4.2 Patterns of Specialization

Predictions

- Let $\Sigma(\omega, \gamma) \equiv \{\sigma \in \Sigma | L(\omega, \sigma, \gamma) > 0\}$ be the set of sectors in which factor ω is employed in country γ
- **Theorem** $\Sigma(\cdot, \cdot)$ *is increasing*
- **Proof:**
  1. Profit maximization $\Rightarrow \Sigma(\omega, \gamma) = \arg\max_{\sigma \in \Sigma} p(\sigma) A(\omega, \sigma, \gamma)$
  2. A1 $\Rightarrow p(\sigma) A(\omega, \sigma, \gamma)$ log-spm by Lemma 1
  3. $p(\sigma) A(\omega, \sigma, \gamma)$ log-spm $\Rightarrow \Sigma(\cdot, \cdot)$ increasing by Lemma 3
- **Corollary** *High-ω factors specialize in high-σ sectors*
- **Corollary** *High-γ countries specialize in high-σ sectors*

4.2 Patterns of Specialization

Relation to the Ricardian literature

- Ricardian model$\equiv$ Special case w/ $A(\omega, \sigma, \gamma) \equiv A(\sigma, \gamma)$
- Previous corollary can help explain:

1. **Multi-country-multi-sector Ricardian model;** Jones (1961)
   - According to Jones (1961), efficient assignment of countries to goods solves $\max \sum \ln A(\sigma, \gamma)$
   - According to Corollary, $A(\sigma, \gamma)$ log-spm implies PAM of countries to goods; Becker (1973), Kremer (1993), Legros and Newman (1996).
2. **Institutions and Trade;** Acemoglu Antras Helpman (2007), Costinot (2006), Cuñat Melitz (2006), Levchenko (2007), Matsuyama (2005), Nunn (2007), and Vogel (2007)
   - Papers vary in terms of source of “institutional dependence” σ and "institutional quality" γ
   - ...but same fundamental objective: providing micro-theoretical foundations for the log-supermodularity of $A(\sigma, \gamma)$

4.3 Aggregate Output, Revenues, and Employment

- Previous results are about the set of goods that each country produces
- **Question:** *Can we say something about how much each country produces? Or how much it employs in each particular sector?*
- **Answer:** *Without further assumptions, the answer is no*

- **A3.** *The profit-maximizing allocation L is unique*
- **A4.** *Factor productivity satisfies* $A(\omega,\sigma,\gamma) \equiv A(\omega,\sigma)$
- **Comments:**
  1. A3 requires $p(\sigma) A(\omega,\sigma,\gamma)$ to be maximized in a *single* sector
  2. A3 is an implicit restriction on the demand-side of the world-economy
     - ... but it becomes milder and milder as the number of factors or countries increases
     - ... generically true if continuum of factors
  3. A4 implies no Ricardian sources of CA across countries
     - Pure Ricardian case can be studied in a similar fashion
     - Having multiple sources of CA is more complex (Costinot 2009)

- **Theorem** *If A3 and 4 hold, then* $Q(\sigma, \gamma)$ *is log-spm.*
- **Proof:**
  1. Let $\Omega(\sigma) \equiv \{\omega \in \Omega | p(\sigma) A(\omega, \sigma) > \max_{\sigma' \neq \sigma} p(\sigma') A(\omega, \sigma')\}$. A3 and A4 imply $Q(\sigma, \gamma) = \int \mathbf{1}_{\Omega(\sigma)}(\omega) \cdot A(\omega, \sigma) V(\omega, \gamma) d\omega$
  2. A1 $\Rightarrow \widetilde{A}(\omega, \sigma) \equiv \mathbf{1}_{\Omega(\sigma)}(\omega) \cdot A(\omega, \sigma)$ log-spm
  3. A2 and $\widetilde{A}(\omega, \sigma)$ log-spm + Lemma 1 $\Rightarrow \widetilde{A}(\omega, \sigma) V(\omega, \gamma)$ log-spm
  4. $\widetilde{A}(\omega, \sigma) V(\omega, \gamma)$ log-spm + Lemma 2 $\Rightarrow Q(\sigma, \gamma)$ log-spm
- **Intuition:**
  1. A1 $\Rightarrow$ high ω-factors are assigned to high σ-sectors
  2. A2 $\Rightarrow$ high ω-factors are more likely in high γ-countries

- **Corollary.** *Suppose that A3 and A4 hold. If two countries produce J goods, with* $\gamma_1 \geq \gamma_2$ *and* $\sigma_1 \geq ... \geq \sigma_J$*, then the high-*$\gamma$ *country tends to specialize in the high-*σ *sectors:*

$$\frac{Q(\sigma_1, \gamma_1)}{Q(\sigma_1, \gamma_2)} \geq ... \geq \frac{Q(\sigma_J, \gamma_1)}{Q(\sigma_J, \gamma_2)}$$

- Let $L(\sigma, \gamma) \equiv \int_{\Omega(\sigma)} V(\omega, \gamma) d\omega$ be aggregate employment
- Let $R(\sigma, \gamma) \equiv \int_{\Omega(\sigma)} r(\omega, \sigma) V(\omega, \gamma) d\omega$ be aggregate revenues
- **Corollary.** *Suppose that A3 and A4 hold. If two countries produce J goods, with $\gamma_1 \geq \gamma_2$ and $\sigma_1 \geq ... \geq \sigma_J$, then aggregate employment and aggregate revenues follow the same pattern as aggregate output:*

$$\frac{L(\sigma_1, \gamma_1)}{L(\sigma_1, \gamma_2)} \geq ... \geq \frac{L(\sigma_J, \gamma_1)}{L(\sigma_J, \gamma_2)} \text{ and } \frac{R(\sigma_1, \gamma_1)}{R(\sigma_1, \gamma_2)} \geq ... \geq \frac{R(\sigma_J, \gamma_1)}{R(\sigma_J, \gamma_2)}$$

4.3 Aggregate Output, Revenues, and Employment

Relation to the previous literature

1. **Worker Heterogeneity and Trade**
   - Generalization of Ruffin (1988):
     - Continuum of factors, Hicks-neutral technological differences
     - Results hold for an arbitrarily large number of goods and factors
   - Generalization of Ohnsorge and Trefler (2007):
     - No functional form assumption (log-normal distribution of human capital, exponential factor productivity)
2. **Firm Heterogeneity and Trade**
   - Closely related to Melitz (2003), Helpman Melitz Yeaple (2004) and Antras Helpman (2004)
     - "Factors" $\equiv$ "Firms" with productivity ω
     - "Countries" $\equiv$ "Industries" with characteristic γ
     - "Sectors" $\equiv$ "Organizations" with characteristic σ
     - $Q(\sigma, \gamma) \equiv$ Sales by firms with "σ-organization" in "γ-industry"
   - In previous papers, $f(\omega, \gamma)$ log-spm is crucial, Pareto is not

5. Comparative Static Predictions

5.1 Closing The Model

Additional assumptions

- Assumptions A1-4 are maintained
- In order to do comparative statics, we also need to specify the demand side of our model:

$$U = \left\{ \int_{\sigma \in \Sigma} \left[C\left(\sigma, \gamma\right) \right]^{\frac{\varepsilon - 1}{\varepsilon}} d\sigma \right\}^{\frac{\varepsilon}{\varepsilon - 1}}$$

- For expositional purposes, we will also assume that:
  - $A\left(\omega, \sigma\right)$ is *strictly* log-supermodular
  - Continuum of factors and sectors: $\Sigma \equiv [\underline{\sigma}, \overline{\sigma}]$ and $\Omega \equiv [\underline{\omega}, \overline{\omega}]$

5.1 Closing the Model

Autarky equilibrium

Autarky equilibrium is a set of functions (Q, C, L, p, w) such that:

1. Firms maximize profit

$$p(\sigma) A(\omega, \sigma) - w(\omega, \gamma) \leq 0, \text{ for all } \omega \in \Omega$$
$$p(\sigma) A(\omega, \sigma) - w(\omega, \gamma) = 0, \text{ for all } \omega \in \Omega \text{ s.t. } L(\omega, \sigma, \gamma) > 0$$

2. Factor markets clear

$$V(\omega, \gamma) = \int_{\sigma \in \Sigma} L(\omega, \sigma, \gamma)\, d\sigma, \text{ for all } \omega \in \Omega$$

3. Consumers maximize their utility and good markets clear

$$C(\sigma, \gamma) = I(\gamma) \times p(\sigma)^{-\varepsilon} = Q(\sigma, \gamma)$$

- **Lemma** *In autarky equilibrium, there exists an increasing bijection $M : \Omega \rightarrow \Sigma$ such that $L(\omega, \sigma) > 0$ if and only if $M(\omega) = \sigma$*
- **Lemma** *In autarky equilibrium, M and w satisfy*

$$\frac{dM(\omega,\gamma)}{d\omega} = \frac{A[\omega, M(\omega,\gamma)]\, V(\omega,\gamma)}{I(\gamma) \times \{p[M(\omega),\gamma]\}^{-\varepsilon}} \tag{1}$$

$$\frac{d \ln w(\omega,\gamma)}{d\omega} = \frac{\partial \ln A[\omega, M(\omega)]}{\partial \omega} \tag{2}$$

with $M(\underline{\omega},\gamma) = \underline{\sigma}$, $M(\overline{\omega},\gamma) = \overline{\sigma}$, and $p[M(\omega,\gamma),\gamma] = w(\omega,\gamma)/A[\omega, M(\omega,\gamma)]$.

5.2 Changes in Factor Supply

- **Question:** *What happens if we change country characteristics from γ to $\gamma' \leq \gamma$?*
- If ω is worker "skill", this can be though of as a change in terms of "skill abundance":

$$\frac{V(\omega,\gamma)}{V(\omega',\gamma)} \geq \frac{V'(\omega,\gamma')}{V'(\omega',\gamma')}, \text{ for all } \omega > \omega'$$

- If $V(\omega,\gamma)$ was a normal distribution, this would correspond to a change in the mean

5.2 Changes in Factor Supply

Consequence for factor allocation

- **Lemma** $M(\omega, \gamma') \geq M(\omega, \gamma)$ *for all* $\omega \in \Omega$
- **Intuition:**
  - If there are relatively more low-ω factors, more sectors should use them
  - From a sector standpoint, this requires *factor downgrading*

- **Proof:** If there is ω s.t. $M(\omega,\gamma') < M(\omega,\gamma)$, then there exist:
  1. $M(\omega_1,\gamma') = M(\omega_1,\gamma) = \sigma_1$, $M(\omega_2,\gamma') = M(\omega_2,\gamma) = \sigma_2$, and $\frac{M_\omega(\omega_1,\gamma')}{M_\omega(\omega_2,\gamma')} \leq \frac{M_\omega(\omega_1,\gamma)}{M_\omega(\omega_2,\gamma)}$
  2. Equation (1) $\Longrightarrow \frac{V(\omega_2,\gamma')}{V(\omega_1,\gamma')}\frac{C(\sigma_1,\gamma')}{C(\sigma_2,\gamma')} \geq \frac{V(\omega_2,\gamma)}{V(\omega_1,\gamma)}\frac{C(\sigma_1,\gamma)}{C(\sigma_2,\gamma)}$
  3. V log-spm $\Longrightarrow \frac{C(\sigma_1,\gamma')}{C(\sigma_2,\gamma')} \geq \frac{C(\sigma_1,\gamma)}{C(\sigma_2,\gamma)}$
  4. Equation (2) + zero profits $\Longrightarrow \frac{d\ln p(\sigma,\gamma)}{d\sigma} = -\frac{\partial \ln A\left[M^{-1}(\sigma,\gamma),\sigma\right]}{\partial\sigma}$
  5. $M^{-1}(\sigma,\gamma) < M^{-1}(\sigma,\gamma')$ for $\sigma \in (\sigma_1,\sigma_2)$ + A log-spm $\Rightarrow \frac{p(\sigma_1,\gamma)}{p(\sigma_2,\gamma)} < \frac{p(\sigma_1,\gamma')}{p'(\sigma_2,\gamma')}$
  6. $\frac{p(\sigma_1,\gamma)}{p(\sigma_2,\gamma)} < \frac{p(\sigma_1,\gamma')}{p'(\sigma_2,\gamma')}$ + CES $\Rightarrow \frac{C(\sigma_1,\gamma')}{C(\sigma_2,\gamma')} > \frac{C(\sigma_1,\gamma)}{C(\sigma_2,\gamma)}$. A contradiction

5.2 Changes in Factor Supply

Consequence for factor prices

- A decrease form γ to γ' implies *pervasive rise in inequality*:

$$\frac{w(\omega, \gamma')}{w(\omega', \gamma')} \geq \frac{w(\omega, \gamma)}{w(\omega', \gamma)}, \text{ for all } \omega > \omega'$$

- The mechanism is simple:
  1. Profit-maximization implies

$$\frac{d \ln w(\omega, \gamma)}{d\omega} = \frac{\partial \ln A[\omega, M(\omega, \gamma)]}{\partial \omega}$$
$$\frac{d \ln w(\omega, \gamma')}{d\omega} = \frac{\partial \ln A[\omega, M(\omega, \gamma')]}{\partial \omega}$$

  2. Since A is log-supermodular, task upgrading implies

$$\frac{d \ln w(\omega, \gamma')}{d\omega} \geq \frac{d \ln w(\omega, \gamma)}{d\omega}$$

- In Costinot Vogel (2010), we also consider changes in diversity
  - This corresponds to the case where there exists $\widehat{\omega}$ such that $V(\omega, \gamma)$ is log-supermodular for $\omega > \widehat{\omega}$, but log-*sub*modular for $\omega < \widehat{\omega}$
- We also consider changes in factor demand (Computerization?):

$$U = \left\{ \int_{\sigma \in \Sigma} B(\sigma, \gamma) \left[C(\sigma, \gamma) \right]^{\frac{\varepsilon - 1}{\varepsilon}} d\sigma \right\}^{\frac{\varepsilon}{\varepsilon - 1}}$$

5.3 North-South Trade

Free trade equilibrium

- Two countries, Home (H) and Foreign (F), with $\gamma_H \geq \gamma_F$
- A competitive equilibrium in the world economy under free trade is s.t.

$$\frac{dM(\omega,\gamma_T)}{d\omega} = \frac{A[\omega, M(\omega,\gamma_T)]\, V(\omega,\gamma_T)}{I_T \times \{p[M(\omega,\gamma_T),\gamma_T]\}^{-\varepsilon}},$$

$$\frac{d\ln w(\omega,\gamma_T)}{d\omega} = \frac{\partial \ln A[\omega, M(\omega,\gamma_T)]}{\partial\omega},$$

where:

$$M(\underline{\omega},\gamma_T) = \underline{\sigma} \text{ and } M(\overline{\omega},\gamma_T) = \overline{\sigma}$$

$$p[M(\omega,\gamma_T),\gamma_T] = w(\omega,\gamma_T)\, A[\omega, M(\omega,\gamma_T)]$$

$$V(\omega,\gamma_T) \equiv V(\omega,\gamma_H) + V(\omega,\gamma_F)$$

- **Key observation:**
$\frac{V(\omega,\gamma_H)}{V(\omega',\gamma_H)} \geq \frac{V(\omega,\gamma_F)}{V(\omega,\gamma_F)}$, for all $\omega > \omega' \Rightarrow \frac{V(\omega,\gamma_H)}{V(\omega',\gamma_H)} \geq \frac{V(\omega,\gamma_T)}{V(\omega',\gamma_T)} \geq \frac{V(\omega,\gamma_F)}{V(\omega,\gamma_F)}$
- Continuum-by-continuum extensions of two-by-two HO results:

1. *Changes in skill-intensities:*

$$M(\omega, \gamma_H) \leq M(\omega, \gamma_T) \leq M(\omega, \gamma_F), \text{ for all } \omega$$

2. *Strong Stolper-Samuelson effect:*

$$\frac{w(\omega, \gamma_H)}{w(\omega', \gamma_H)} \leq \frac{w(\omega, \gamma_T)}{w(\omega', \gamma_T)} \leq \frac{w(\omega, \gamma_F)}{w(\omega', \gamma_F)}, \text{ for all } \omega > \omega'$$

- North-South trade driven by factor demand differences:
  - Same logic gets to the exact opposite results
  - Correlation between factor demand and factor supply considerations matters
- One can also extend analysis to study “North-North” trade:
  - It predicts wage polarization in the more diverse country and wage convergence in the other

What's next?

- Dynamic issues:
  - Sector-specific human capital accumulation
  - Endogenous technology adoption
- Empirics:
  - Revisiting the consequences of trade liberalization