

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
 Department of Electrical Engineering & Computer Science  
**6.041/6.431: Probabilistic Systems Analysis**  
 (Final Exam Solutions | Spring 2009)

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**Problem 1: True or False** (2pts. each, 18 pts. total)

**No partial credit** will be given for individual questions in this part of the quiz.

a. Let  $\{X_n\}$  be a sequence of i.i.d random variables taking values in the interval  $[0, 0.5]$ . Consider the following statements:

- (A) If  $\mathbf{E}[X_n^2]$  converges to 0 as  $n \rightarrow \infty$  then  $X_n$  converges to 0 in probability.
- (B) If all  $X_n$  have  $\mathbf{E}[X_n] = 0.2$  and  $\text{var}(X_n)$  converges to 0 as  $n \rightarrow \infty$  then  $X_n$  converges to 0.2 in probability.
- (C) The sequence of random variables  $Z_n$ , defined by  $Z_n = X_1 \cdot X_2 \cdots X_n$ , converges to 0 in probability as  $n \rightarrow \infty$ .

Which of these statements are **always** true? Write **True** or **False** in each of the boxes below.

A: True	B: True	C: True
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**Solution:**

- (A) True. The fact that  $\lim_{n \rightarrow \infty} \mathbf{E}[X_n^2] = 0$  implies  $\lim_{n \rightarrow \infty} \mathbf{E}[X_n] = 0$  and  $\lim_{n \rightarrow \infty} \text{var}(X_n) = 0$ . Hence, one has

$$\begin{aligned} \mathbf{P}(|X_n - 0| \geq \epsilon) &\leq \mathbf{P}(|X_n - \mathbf{E}[X_n]| \geq \epsilon/2) + \mathbf{P}(|\mathbf{E}[X_n] - 0| \geq \epsilon/2) \\ &\leq \frac{\text{var}(X_n)}{(\epsilon/2)^2} + \mathbf{P}(|\mathbf{E}[X_n] - 0| \geq \epsilon/2) \rightarrow 0, \end{aligned}$$

where we have applied Chebyshev inequality.

- (B) True. Applying Chebyshev inequality gives

$$\mathbf{P}(|X_n - \mathbf{E}[X_n]| \geq \epsilon) \leq \frac{\text{var}(X_n)}{\epsilon^2} \rightarrow 0.$$

Hence  $X_n$  converges to  $\mathbf{E}[X_n] = 0.2$  in probability.

- (C) True. For all  $\epsilon > 0$ , since  $Z_n \leq (1/2)^n \Rightarrow \mathbf{P}(|Z_n - 0| \geq \epsilon) = 0$  for  $n > -\log \epsilon / \log 2$ .

b. Let  $X_i$  ( $i = 1, 2, \dots$ ) be i.i.d. random variables with mean 0 and variance 2;  $Y_i$  ( $i = 1, 2, \dots$ ) be i.i.d. random variables with mean 2. Assume that all variables  $X_i, Y_j$  are independent. Consider the following statements:

- (A)  $\frac{X_1 + \dots + X_n}{n}$  converges to 0 in probability as  $n \rightarrow \infty$ .
- (B)  $\frac{X_1^2 + \dots + X_n^2}{n}$  converges to 2 in probability as  $n \rightarrow \infty$ .
- (C)  $\frac{X_1 Y_1 + \dots + X_n Y_n}{n}$  converges to 0 in probability as  $n \rightarrow \infty$ .

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Which of these statements are **always** true? Write **True** or **False** in each of the boxes below.

A: True	B: True	C: True
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**Solution:**

- (A) True. Note that  $\mathbf{E}[\frac{X_1+\dots+X_n}{n}] = 0$  and  $\text{var}(\frac{X_1+\dots+X_n}{n}) = \frac{n \cdot 2}{n^2} = \frac{2}{n}$ . One can see  $\frac{X_1+\dots+X_n}{n}$  converges to 0 in probability.
- (B) True. Let  $Z_i = X_i^2$  and  $\mathbf{E}[Z_i] = 2$ . Note  $Z_i$  are i.i.d. since  $X_i$  are i.i.d., and hence one has that  $\frac{Z_1+\dots+Z_n}{n}$  converges to  $\mathbf{E}[Z_i] = 2$  in probability by the WLLN.
- (C) True. Let  $W_i = X_i Y_i$  and  $\mathbf{E}[W_i] = \mathbf{E}[X_i] \mathbf{E}[Y_i] = 0$ . Note  $W_i$  are i.i.d. since  $X_i$  and  $Y_i$  are respectively i.i.d., and hence one has that  $\frac{W_1+\dots+W_n}{n}$  converges to  $\mathbf{E}[W_i] = 0$  in probability by the WLLN.

c. We have i.i.d. random variables  $X_1 \dots X_n$  with an unknown distribution, and with  $\mu = \mathbf{E}[X_i]$ . We define  $M_n = (X_1 + \dots + X_n)/n$ . Consider the following statements:

- (A)  $M_n$  is a maximum-likelihood estimator for  $\mu$ , irrespective of the distribution of the  $X_i$ 's.
- (B)  $M_n$  is a consistent estimator for  $\mu$ , irrespective of the distribution of the  $X_i$ 's.
- (C)  $M_n$  is an asymptotically unbiased estimator for  $\mu$ , irrespective of the distribution of the  $X_i$ 's.

Which of these statements are **always** true? Write **True** or **False** in each of the boxes below.

A: False	B: True	C: True
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**Solution:**

- (A) False. Consider  $X_i$  follow a uniform distribution  $U[\mu - \frac{1}{2}, \mu + \frac{1}{2}]$ . The ML estimator for  $\mu$  is any value between  $\max(X_1, \dots, X_n) - \frac{1}{2}$  and  $\min(X_1, \dots, X_n) + \frac{1}{2}$ , instead of  $M_n$ .
- (B) True. By the WLLN,  $M_n$  converges to  $\mu$  in probability and hence it is a consistent estimator.
- (C) True. Since  $\mathbf{E}[M_n] = \mathbf{E}[X_i] = \mu$ ,  $M_n$  is unbiased estimator for  $\mu$  and hence asymptotically unbiased.

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**Problem 2: Multiple Choice** (4 pts. each, 24 pts. total)

Clearly circle the appropriate choice. No partial credit will be given for individual questions in this part of the quiz.

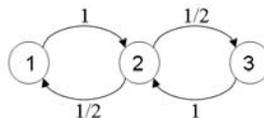
- a. Earthquakes in Sumatra occur according to a Poisson process of rate  $\lambda = 2/\text{year}$ . Conditioned on the event that exactly two earthquakes take place in a year, what is the probability that both earthquakes occur in the first three months of the year? (for simplicity, assume all months have 30 days, and each year has 12 months, i.e., 360 days).
- (i)  $1/12$
  - (ii)  $1/16$
  - (iii)  $64/225$
  - (iv)  $4e^{-4}$
  - (v) There is not enough information to determine the required probability.
  - (vi) None of the above.

**Solution:** Consider the interval of a year be  $[0, 1]$ .

$$\begin{aligned} \mathbf{P}\left(2 \text{ in } \left[0, \frac{1}{4}\right) \mid 2 \text{ in } [0, 1]\right) &= \frac{\mathbf{P}\left(2 \text{ in } \left[0, \frac{1}{4}\right), 0 \text{ in } \left[\frac{1}{4}, 1\right]\right)}{\mathbf{P}(2 \text{ in } [0, 1])} \\ &= \frac{\frac{(\lambda \cdot 1/4)^2}{2!} e^{-\lambda \cdot 1/4} \cdot \frac{(\lambda \cdot 3/4)^0}{0!} e^{-\lambda \cdot 3/4}}{\frac{\lambda^2}{2!} e^{-\lambda}} \\ &= \frac{1}{16} \end{aligned}$$

(alternative explanation) Given that exactly two earthquakes happened in 12 months, each earthquake is equally likely to happen in any month of the 12, the probability that it happens in the first 3 months is  $3/12 = 1/4$ . The probability that both happen in the first 3 months is  $(1/4)^2$ .

- b. Consider a continuous-time Markov chain with three states  $i \in \{1, 2, 3\}$ , with dwelling time in each visit to state  $i$  being an exponential random variable with parameter  $\nu_i = i$ , and transition probabilities  $p_{ij}$  defined by the graph



What is the long-term expected fraction of time spent in state 2?

- (i)  $1/2$
- (ii)  $1/4$
- (iii)  $2/5$

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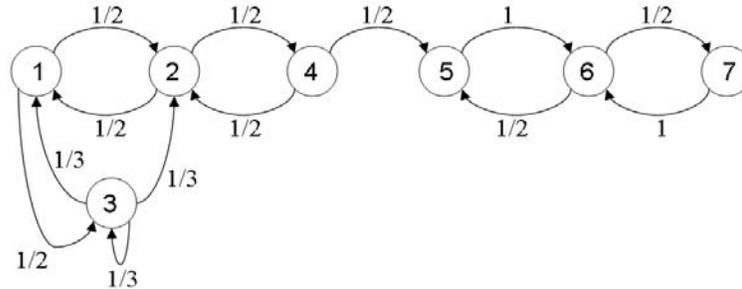
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(iv)  $\boxed{3/7}$

(v) None of the above.

**Solution:** First, we calculate the  $q_{ij} = \nu_i p_{ij}$ , i.e.,  $q_{12} = q_{21} = q_{23} = 1$  and  $q_{32} = 3$ . The balance and normalization equations of this birth-death markov chain can be expressed as,  $\pi_1 = \pi_2$ ,  $\pi_2 = 3\pi_3$  and  $\pi_1 + \pi_2 + \pi_3 = 1$ , yielding  $\pi_2 = 3/7$ .

c. Consider the following Markov chain:



Starting in state 3, what is the steady-state probability of being in state 1?

- (i) 1/3
- (ii) 1/4
- (iii) 1
- (iv)  0
- (v) None of the above.

**Solution:** State 1 is transient.

d. Random variables  $X$  and  $Y$  are such that the pair  $(X, Y)$  is uniformly distributed over the trapezoid  $A$  with corners  $(0, 0)$ ,  $(1, 2)$ ,  $(3, 2)$ , and  $(4, 0)$  shown in Fig. 1:

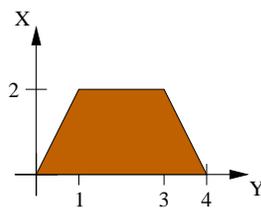


Figure 1:  $f_{X,Y}(x, y)$  is constant over the shaded area, zero otherwise.

i.e.

$$f_{X,Y}(x, y) = \begin{cases} c, & (x, y) \in A \\ 0, & \text{else.} \end{cases}$$

We observe  $Y$  and use it to estimate  $X$ . Let  $\hat{X}$  be the least mean squared error estimator of  $X$  given  $Y$ . What is the value of  $\text{var}(\hat{X} - X|Y = 1)$ ?

- (i) 1/6
- (ii) 3/2
- (iii)  1/3
- (iv) The information is not sufficient to compute this value.

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(v) None of the above.

**Solution:**  $f_{X|Y=1}(x)$  is uniform on  $[0, 2]$  therefore  $\hat{X} = \mathbf{E}[X|Y = 1] = 1$  and  $\text{var}(\hat{X} - X|Y = 1) = \text{var}(X|Y = 1) = (2 - 0)^2/12 = 1/3$ .

e.  $X_1 \dots X_n$  are i.i.d. normal random variables with mean value  $\mu$  and variance  $v$ . Both  $\mu$  and  $v$  are unknown. We define  $M_n = (X_1 + \dots + X_n)/n$  and

$$V_n = \frac{1}{n-1} \sum_{i=1}^n (X_i - M_n)^2$$

We also define  $\Phi(x)$  to be the CDF for the standard normal distribution, and  $\Psi_{n-1}(x)$  to be the CDF for the t-distribution with  $n - 1$  degrees of freedom. Which of the following choices gives an exact 99% confidence interval for  $\mu$  for all  $n > 1$ ?

(i)  $[M_n - \delta\sqrt{\frac{V_n}{n}}, M_n + \delta\sqrt{\frac{V_n}{n}}]$  where  $\delta$  is chosen to give  $\Phi(\delta) = 0.99$ .

(ii)  $[M_n - \delta\sqrt{\frac{V_n}{n}}, M_n + \delta\sqrt{\frac{V_n}{n}}]$  where  $\delta$  is chosen to give  $\Phi(\delta) = 0.995$ .

(iii)  $[M_n - \delta\sqrt{\frac{V_n}{n}}, M_n + \delta\sqrt{\frac{V_n}{n}}]$  where  $\delta$  is chosen to give  $\Psi_{n-1}(\delta) = 0.99$ .

(iv)  $[M_n - \delta\sqrt{\frac{V_n}{n}}, M_n + \delta\sqrt{\frac{V_n}{n}}]$  where  $\delta$  is chosen to give  $\Psi_{n-1}(\delta) = 0.995$ .

(v) None of the above.

**Solution:** See Lecture 23, slides 10-12.

f. We have i.i.d. random variables  $X_1, X_2$  which have an exponential distribution with unknown parameter  $\theta$ . Under hypothesis  $H_0$ ,  $\theta = 1$ . Under hypothesis  $H_1$ ,  $\theta = 2$ . Under a likelihood-ratio test, the rejection region takes which of the following forms?

(i)  $R = \{(x_1, x_2) : x_1 + x_2 > \xi\}$  for some value  $\xi$ .

(ii)  $R = \{(x_1, x_2) : x_1 + x_2 < \xi\}$  for some value  $\xi$ .

(iii)  $R = \{(x_1, x_2) : e^{x_1} + e^{x_2} > \xi\}$  for some value  $\xi$ .

(iv)  $R = \{(x_1, x_2) : e^{x_1} + e^{x_2} < \xi\}$  for some value  $\xi$ .

(v) None of the above.

**Solution:** We defined  $R = \{x = (x_1, x_2) | L(x) > c\}$  where

$$L(x) = \frac{f_X(x; H_1)}{f_X(x; H_0)} = \frac{\theta_1 e^{-\theta_1 x_1} \theta_1 e^{-\theta_1 x_2}}{\theta_0 e^{-\theta_0 x_1} \theta_0 e^{-\theta_0 x_2}} = \frac{\theta_1^2}{\theta_0^2} e^{(\theta_0 - \theta_1)(x_1 + x_2)} = 4e^{-(x_1 + x_2)}.$$

So  $R = \{(x_1, x_2) | x_1 + x_2 < -\log(c/4)\}$

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**Problem 3** (12 pts. total)

Aliens of two races (blue and green) are arriving on Earth independently according to Poisson process distributions with parameters  $\lambda_b$  and  $\lambda_g$  respectively. The Alien Arrival Registration Service Authority (AARSA) will begin registering alien arrivals soon.

Let  $T_1$  denote the time AARSA will function until it registers its first alien. Let  $G$  be the event that the first alien to be registered is a green one. Let  $T_2$  be the time AARSA will function until at least one alien of both races is registered.

- (a) (4 points.) Express  $\mu_1 = \mathbf{E}[T_1]$  in terms of  $\lambda_g$  and  $\lambda_b$ . **Show your work.**

**Answer:**  $\mu_1 = \mathbf{E}[T_1] = \frac{1}{\lambda_g + \lambda_b}$

**Solution:** We consider the process of arrivals of both types of Aliens. This is a merged Poisson process with arrival rate  $\lambda_g + \lambda_b$ .  $T_1$  is the time until the first arrival, and therefore is exponentially distributed with parameter  $\lambda_g + \lambda_b$ . Therefore  $\mu_1 = \mathbf{E}[T_1] = \frac{1}{\lambda_g + \lambda_b}$ .

One can also go about this using derived distributions, since  $T_1 = \min(T_1^g, T_1^b)$  where  $T_1^g$  and  $T_1^b$  are the first arrival times of green and blue Aliens respectively (i.e.,  $T_1^g$  and  $T_1^b$  are exponentially distributed with parameters  $\lambda_g$  and  $\lambda_b$ , respectively. )

- (b) (4 points.) Express  $p = \mathbf{P}(G)$  in terms of  $\lambda_g$  and  $\lambda_b$ . **Show your work.**

**Answer:**  $\mathbf{P}(G) = \frac{\lambda_g}{\lambda_g + \lambda_b}$

**Solution:** We consider the same merged Poisson process as before, with arrival rate  $\lambda_g + \lambda_b$ . Any particular arrival of the merged process has probability  $\frac{\lambda_g}{\lambda_g + \lambda_b}$  of corresponding to a green Alien and probability  $\frac{\lambda_b}{\lambda_g + \lambda_b}$  of corresponding to a blue Alien. The question asks for  $\mathbf{P}(G) = \frac{\lambda_g}{\lambda_g + \lambda_b}$ .

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- (c) (4 points.) Express  $\mu_2 = \mathbf{E}[T_2]$  in terms of  $\lambda_g$  and  $\lambda_b$ .  
**Show your work.**

<b>Answer:</b> $\frac{1}{\lambda_g + \lambda_b} + \frac{\lambda_g}{\lambda_g + \lambda_b} \left(\frac{1}{\lambda_b}\right) + \frac{\lambda_b}{\lambda_g + \lambda_b} \left(\frac{1}{\lambda_g}\right)$
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**Solution:** The time  $T_2$  until at least one green and one red Aliens have arrived can be expressed as  $T_2 = \max(T_1^g, T_1^b)$ , where  $T_1^g$  and  $T_1^b$  are the first arrival times of green and blue Aliens respectively (i.e.,  $T_1^g$  and  $T_1^b$  are exponentially distributed with parameters  $\lambda_g$  and  $\lambda_b$ , respectively.)

The expected time till the 1st Alien arrives was calculated in (a),  $\mu_1 = \mathbf{E}[T_1] = \frac{1}{\lambda_g + \lambda_b}$ . To compute the remaining time we simply condition on the 1st Alien being green (e.g. event  $G$ ) or blue (event  $G^c$ ), and use the memoryless property of Poisson, i.e.,

$$\begin{aligned} \mathbf{E}[T_2] &= \mathbf{E}[T_1] + \mathbf{P}(G)\mathbf{E}[\text{Time until first Blue arrives}|G] + \mathbf{P}(G^c)\mathbf{E}[\text{Time until first Green arrives}|G^c] \\ &= \mathbf{E}[T_1] + \mathbf{P}(G)\mathbf{E}[T_2^b] + (1 - \mathbf{P}(G))\mathbf{E}[T_2^g] \\ &= \frac{1}{\lambda_g + \lambda_b} + \frac{\lambda_g}{\lambda_g + \lambda_b} \left(\frac{1}{\lambda_b}\right) + \frac{\lambda_b}{\lambda_g + \lambda_b} \left(\frac{1}{\lambda_g}\right) \end{aligned}$$

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**Problem 4** (18 pts. total)

Researcher Jill is interested in studying employment in technology firms in Dilicon Valley. She denotes by  $X_i$  the number of employees in technology firm  $i$  and assumes that  $X_i$  are independent and identically distributed with mean  $p$ . To estimate  $p$ , Jill randomly interviews  $n$  technology firms and observes the number of employees in these firms.

(a) (6 points.) Jill uses

$$M_n = \frac{X_1 + \cdots + X_n}{n}$$

as an estimator for  $p$ . Find the limit of  $\mathbf{P}(M_n \leq x)$  as  $n \rightarrow \infty$  for  $x < p$ . Find the limit of  $\mathbf{P}(M_n \leq x)$  as  $n \rightarrow \infty$  for  $x > p$ . **Show your work.**

**Solution:** Since  $X_i$  is i.i.d.,  $M_n$  converges to  $p$  in probability, i.e.,  $\lim_{n \rightarrow \infty} \mathbf{P}(|M_n - p| > \epsilon) = 0$ , implying  $\lim_{n \rightarrow \infty} \mathbf{P}(M_n < p - \epsilon) = 0$  and  $\lim_{n \rightarrow \infty} \mathbf{P}(M_n > p + \epsilon) = 0$ , for all  $\epsilon > 0$ . Hence

$$\lim_{n \rightarrow \infty} \mathbf{P}(M_n \leq x) = \begin{cases} 0, & x < p, \\ 1, & x > p. \end{cases}$$

(b) (6 points.) Find the smallest  $n$ , the number of technology firms Jill must sample, for which the Chebyshev inequality yields a guarantee

$$\mathbf{P}(|M_n - p| \geq 0.5) \leq 0.05.$$

Assume that  $\text{var}(X_i) = v$  for some constant  $v$ . State your solution as a function of  $v$ . **Show your work.**

**Solution:** Since  $M_n$  converges to  $p$  in probability and  $\text{var}(M_n) = \frac{n}{n^2} \cdot \text{var}(X_i) = v/n$ , Chebyshev inequality gives

$$P(|M_n - p| \geq 0.5) \leq \frac{\text{var}(M_n)}{0.5^2} = \frac{v}{n \cdot 0.5^2} = 0.05$$

$$\Rightarrow \boxed{n = 80v.}$$

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- (c) (6 points.) Assume now that the researcher samples  $n = 5000$  firms. Find an approximate value for the probability

$$\mathbf{P}(|M_{5000} - p| \geq 0.5)$$

using the Central Limit Theorem. Assume again that  $\text{var}(X_i) = v$  for some constant  $v$ . Give your answer in terms of  $v$ , and the standard normal CDF  $\Phi$ . **Show your work.**

**Solution:** By CLT, we can approximate by a standard normal distribution

$$\frac{\sum_{i=1}^n X_i - np}{\sqrt{nv}}$$

when  $n$  is large, and hence,

$$\mathbf{P}(|M_{5000} - p| \geq 0.5) = P\left(\left|\frac{\sum_{i=1}^n X_i - np}{\sqrt{nv}}\right| \geq \frac{0.5\sqrt{n}}{\sqrt{v}}\right) = \boxed{2 - 2\Phi\left(\frac{0.5\sqrt{n}}{\sqrt{v}}\right)},$$

where  $n = 5000$ .

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**Problem 5** (12 pts. total)

The RandomView window factory produces window panes. After manufacturing, 1000 panes were loaded onto a truck. The weight  $W_i$  of the  $i$ -th pane (in pounds) on the truck is modeled as a random variable, with the assumption that the  $W_i$ 's are independent and identically distributed.

- (a) (6 points.) Assume that the measured weight of the load on the truck was 2340 pounds, and that  $\text{var}(W_i) \leq 4$ . Find an approximate 95 percent confidence interval for  $\mu = \mathbf{E}[W_i]$ , using the Central Limit Theorem (you may use the standard normal table which was handed out with this quiz). **Show your work.**

**Answer:** [2.216, 2.464]

**Solution:** The sample mean estimator  $\hat{\Theta}_n = \frac{W_1 + \dots + W_n}{n}$  in this case is

$$\hat{\Theta}_{1000} = \frac{2340}{1000} = 2.34$$

Using the CDF  $\Phi(z)$  of the standard normal available in the normal tables, we have  $\Phi(1.96) = 0.975$ , so we obtain

$$\mathbf{P}\left(\frac{|\hat{\Theta}_{1000} - \mu|}{\sqrt{\text{var}(W_i)/1000}} \leq 1.96\right) \approx 0.95.$$

Because the variance is less than 4, we have

$$\mathbf{P}(|\hat{\Theta}_{1000} - \mu| \leq 1.96\sqrt{\text{var}(W_i)/1000}) \leq \mathbf{P}(|\hat{\Theta}_{1000} - \mu| \leq 1.96\sqrt{4/1000}),$$

and letting the right-hand side of the above equation  $\approx 0.95$  gives a 95% confidence, i.e.,

$$\left[\hat{\Theta}_{1000} - 1.96\sqrt{\frac{4}{1000}}, \hat{\Theta}_{1000} + 1.96\sqrt{\frac{4}{1000}}\right] = [\hat{\Theta}_{1000} - 0.124, \hat{\Theta}_{1000} + 0.124]$$

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- (b) (6 points.) Now assume instead that the random variables  $W_i$  are i.i.d, with an exponential distribution with parameter  $\theta > 0$ , i.e., a distribution with PDF

$$f_W(w; \theta) = \theta e^{-\theta w}$$

What is the maximum likelihood estimate of  $\theta$ , given that the truckload has weight 2340 pounds?  
**Show your work.**

<b>Answer:</b> $\hat{\Theta}_{1000}^{mle} = \frac{1000}{2340} = 0.4274$
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**Solution:** The likelihood function is

$$f_W(w; \theta) = \prod_{i=1}^n f_{W_i}(w_i; \theta) = \prod_{i=1}^n \theta e^{-\theta w_i},$$

And the log-likelihood function is

$$\log f_W(w; \theta) = n \log \theta - \theta \sum_{i=1}^n w_i,$$

The derivative with respect to  $\theta$  is  $\frac{n}{\theta} - \sum_{i=1}^n w_i$ , and by setting it to zero, we see that the maximum of  $\log f_W(w; \theta)$  over  $\theta \geq 0$  is attained at  $\hat{\theta}_n = \frac{n}{\sum_{i=1}^n w_i}$ . The resulting estimator is

$$\hat{\Theta}_n^{mle} = \frac{n}{\sum_{i=1}^n W_i}.$$

In our case,

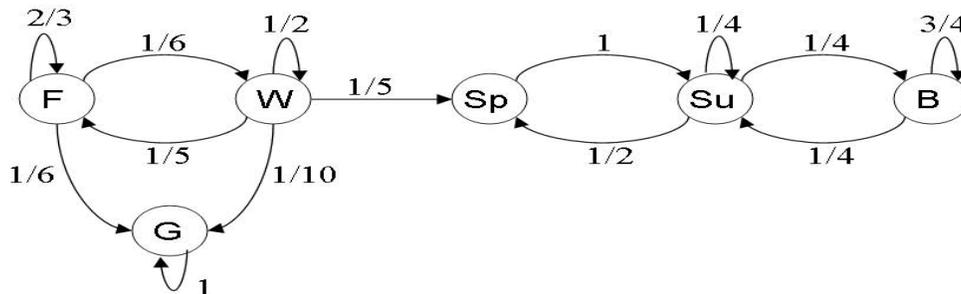
$$\hat{\Theta}_{1000}^{mle} = \frac{1000}{2340} = 0.4274$$

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**Problem 6** (21 pts. total)

In Alice's Wonderland, there are six different seasons: Fall (F), Winter (W), Spring (Sp), Summer (Su), Bitter Cold (B), and Golden Sunshine (G). The seasons do not follow any particular order, instead, at the beginning of each day the Head Wizard assigns the season for the day, according to the following Markov chain model:



Thus, for example, if it is Fall one day then there is 1/6 probability that it will be Winter the next day (note that it is possible to have the same season again the next day).

- (a) (4 points.) For each state in the above chain, identify whether it is recurrent or transient. **Show your work.**

**Solution:** F and W are transient states; Sp, Su, B, and G are recurrent states.

- (b) (4 points.) If it is Fall on Monday, what is the probability that it will be Summer on Thursday of the same week? **Show your work.**

**Solution:** There is only one path from F to Su in three days.

$$\begin{aligned}
 \mathbf{P}(S_4 = \text{Su} | S_1 = \text{F}) &= \mathbf{P}(S_2 = \text{W} | S_1 = \text{F}) \cdot \mathbf{P}(S_3 = \text{Sp} | S_2 = \text{W}) \cdot \mathbf{P}(S_4 = \text{Su} | S_3 = \text{Sp}) \\
 &= \frac{1}{6} \cdot \frac{1}{5} \cdot 1 = \boxed{\frac{1}{30}}
 \end{aligned}$$

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- (c) (4 points.) If it is Spring today, will the chain converge to steady-state probabilities? If so, compute the steady-state probability for each state. If not, explain why these probabilities do not exist. **Show your work.**

**Solution:** The Markov chain will stay in the recurrent class  $\{Sp, Su, B\}$ , and

$$\begin{cases} \pi_{Sp} \cdot 1 = \pi_{Su} \cdot \frac{1}{2} \\ \pi_B \cdot \frac{1}{4} = \pi_{Su} \cdot \frac{1}{4} \\ \pi_F = 0 \\ \pi_W = 0 \\ \pi_G = 0 \\ \pi_F + \pi_W + \pi_G + \pi_{Sp} + \pi_{Su} + \pi_B = 1 \end{cases}$$

$$\Rightarrow \boxed{\pi_F = 0, \pi_W = 0, \pi_G = 0, \pi_{Sp} = 1/5, \pi_{Su} = 2/5, \pi_B = 2/5.}$$

- (d) (5 points.) If it is Fall today, what is the probability that Bitter Cold will never arrive in the future? **Show your work.**

**Solution:** Let  $a_F$  and  $a_W$  be the probabilities that Bitter Cold will never arrive starting from Fall and Winter, respectively. This is equivalent to the Markov chain ends up in G.

$$\begin{cases} a_F = \frac{2}{3} \cdot a_F + \frac{1}{6} \cdot a_W + \frac{1}{6} \cdot 1 \\ a_W = \frac{1}{5} \cdot a_F + \frac{1}{2} \cdot a_W + \frac{1}{10} \cdot 1 \end{cases}$$

$$\Rightarrow \boxed{a_F = 3/4.}$$

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- (e) (5 points.) If it is Fall today, what is the expected number of days till either Summer or Golden Sunshine arrives for the first time? **Show your work.**

**Solution:** Let  $\mu_F$  and  $\mu_W$  be expected number of days till either Summer or Golden Sunshine arrives for the first time, respectively.

$$\begin{cases} \mu_F = 1 + \frac{2}{3} \cdot \mu_F + \frac{1}{6} \cdot \mu_W + \frac{1}{6} \cdot 0 \\ \mu_W = 1 + \frac{1}{5} \cdot \mu_F + \frac{1}{2} \cdot \mu_W + \frac{1}{5} \cdot 1 \end{cases}$$

$\Rightarrow$   $\mu_F = 5.25.$

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**Problem 7** (12 pts. total)

A newscast covering the final baseball game between Sed Rox and Y Nakee becomes noisy at the crucial moment when the viewers are informed whether Y Nakee won the game.

Let  $a$  be the parameter describing the actual outcome:  $a = 1$  if Y Nakee won,  $a = -1$  otherwise. There were  $n$  viewers listening to the telecast. Let  $Y_i$  be the information received by viewer  $i$  ( $1 \leq i \leq n$ ). Under the noisy telecast,  $Y_i = a$  with probability  $p$ , and  $Y_i = -a$  with probability  $1 - p$ . Assume that the random variables  $Y_i$  are independent of each other.

The viewers as a group come up with a joint estimator

$$Z_n = \begin{cases} 1 & \text{if } \sum_{i=1}^n Y_i \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

- (a) (6 points.) Find  $\lim_{n \rightarrow \infty} \mathbf{P}(Z_n = a)$  assuming that  $p > 0.5$  and  $a = 1$ . **Show your work.**

**Solution:** Note that

$$\lim_{n \rightarrow \infty} \mathbf{P}(Z_n = 1) = \lim_{n \rightarrow \infty} \mathbf{P}\left(\sum_{i=1}^n Y_i \geq 0\right) = \lim_{n \rightarrow \infty} \mathbf{P}\left(\frac{\sum_{i=1}^n Y_i}{n} \geq 0\right).$$

Since  $Y_i$  are i.i.d. with mean  $\mathbf{E}[Y_i] = 2p - 1$  and finite variance  $\text{var}(Y_i) = 1 - (2p - 1)^2$ , one has, by Chebyshev inequality, for all  $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(\left|\frac{\sum_{i=1}^n Y_i}{n} - (2p - 1)\right| \geq \epsilon\right) = 0.$$

Take  $\epsilon = p - \frac{1}{2}$ , and the above equation implies  $\lim_{n \rightarrow \infty} \mathbf{P}\left(\frac{\sum_{i=1}^n Y_i}{n} \leq (2p - 1)/2\right) = 0$ . Therefore,

$$\boxed{\lim_{n \rightarrow \infty} \mathbf{P}(Z_n = 1) = 1.}$$

- (b) (6 points.) Find  $\lim_{n \rightarrow \infty} \mathbf{P}(Z_n = a)$ , assuming that  $p = 0.5$  and  $a = 1$ . **Show your work.**

**Solution:** Note that

$$\lim_{n \rightarrow \infty} \mathbf{P}(Z_n = 1) = \lim_{n \rightarrow \infty} \mathbf{P}\left(\sum_{i=1}^n Y_i \geq 0\right) = \lim_{n \rightarrow \infty} \mathbf{P}\left(\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \geq 0\right).$$

Since  $Y_i$  are i.i.d. with  $\mathbf{E}[Y_i] = 0$  and  $\text{var}(Y_i) = 1$ , we can approximate  $\frac{\sum_{i=1}^n Y_i}{\sqrt{n}}$  as a standard normal random variable when  $n$  goes to infinity. Thus,  $\boxed{\lim_{n \rightarrow \infty} \mathbf{P}(Z_n = 1) = 1/2.}$

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