6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Tutorial: The Probability Distribution Function (PDF) of [X]

Hi, In this problem, we'll be looking at the PDF the absolute value of x. So if we know a random variable, x, and we know it's PDF, how can we use that information to help us find the PDF of another random variable-- the absolute value of x? And so throughout this problem, we'll define a new random variable called y. And we'll define that y to be equal to the absolute value of x, just to make things simpler.

So we'll do a couple of concrete examples, and then we'll try to generalize at the end. The first example that we'll deal with in part A is this PDF for x. So we're told that the PDF of x is 1/3 between negative 2 and 1, and 0 otherwise. And here's a picture of what it looks like. It's just a rectangle from negative 2 to 1. So now we want to find out what is the PDF of the absolute value of x, which we've called y?

And at this point, it may be helpful to step back and think about this problem from the discrete point of view again. So if x were a discrete random variable, the problem would be, what is the probability that the absolute value of x is equal to, say, 1/2? Well, the probability that the absolute value of x is equal to 1/2-- that can occur in two different ways. One is that x itself is 1/2. Or x could be negative 1/2, in which case, the absolute value of x would still be 1/2.

So those two events are mutually exclusive. And so the probability of either one of them happening is you can just add them up. And so the probability of the absolute value of x being 1/2 would have two contributions, one from x being 1/2, and one from x being negative 1/2.

The analogous idea carries over to the continuous case, when you have a PDF. So now let's say that we're interested in the case where we want to know the PDF of y at 1/2. Well, that again, is going to have two contributions, one from where x is 1/2, and one from where x is minus 1/2. And so you can just imagine that each one of these values for y-- and remember, y has to be non-negative, because it's an absolute value-- has two contributions, one from the right side of 0, and one from the left, or negative, side of 0.

So you can come up and write an algebraic expression for this, and we'll do that in Part C. But you can also look at this from a visual point of view. And you can take the PDF diagram itself and imagine transforming it to find out what the PDF of the absolute value of x would look like. So the way to do it would be you take what's on the negative side. You flip it over and take the mirror image, and then you stack it on top of what you have on the right-hand side, or the positive side.

So take this, flip it over, and stack it on top. You can imagine just taking this block, flipping it over. And just think of it as like a Tetris block that's falling down from above. And it stacks on top of wherever it lands. So it'll turn it into something that looks like this.

So there's already a block of height 1/3 from 0 to 1. That's from the original 1. And now we take this, and flip it over, and drop it on top. Well, this part is going to fall on top of the segment from 0 to 1.

And then this part gets flipped over and dropped over here. And it falls down here. And so the final PDF actually looks like this kind of staircase, where this is 2/3 now, because this has two contributions of 1/3 each, and this is 1/3.

So that is the graphical way of approaching this. And the PDF for completeness, the PDF of y would be 2/3 for y between 0 and 1, 1/3 for y from 1 to 2, and 0 otherwise.

All right, so let's move on to part B, and get some more practice. Part B, we're given that this PDF of x now is 2 times e to the negative 2x for x positive, and 0 otherwise. Now you may just recognize this as an exponential random variable with a parameter of 2. And again, we can graph this and see what it looks like. And it turns out that it's going to start out at 2 and fall off exponentially.

So in this case, this is actually quite simple. Because if you look at it, x is already positive. It doesn't have any negative parts. So in fact, the absolute value of x is the same as x itself, because x is never negative.

And so y is just the same thing as x. And so in this case, actually, the PDF of y is exactly the same as the PDF of x. It's just 2e to the minus 2y, for y positive and zero otherwise.

Now you can also see this graphically also, because to the left of 0, the negative part, there is no PDF. The PDF is 0. And so if you were to take this, flip it over, and drop it on top, you wouldn't get anything, because there's nothing there. And so the entire PDF, even after you take the absolute value, is just the original one.

So to generalize, what I said at the beginning was that, remember, the probability in the discrete case, if you wanted the probability that the absolute value of a random variable equals something, that would just be the probability that the random variable equals that value of little x, or the random variable equals negative little x.

In either of those two cases, the absolute value would equal x. So you get those two contributions. And so to generalize in the continuous case with PDFs, you get something that looks very similar. So in this case, the PDF or y is just the PDF of x at y.

So this is the case where x is just equal to y, plus the PDF of x evaluated negative y. So you, again, have both of these two contributions. And we can rewrite this top one to make it look more similar. So the PMF of some discrete [? number ?] y, where this is a discrete random variable that's equal to the absolute value of x, would be the PMF of x evaluated at y, plus the PMF of x evaluated at negative y. So in both the discrete and continuous cases, you have the same thing.

So the overall summary of this problem is that, when you take a transformation-- in this case, an absolute value-- you can reason about it and figure out how to decompose that into arguments about the original random variable, just plain old x. And for the specific case of the absolute value, it just becomes taking a mirror image and popping it on top of what you originally had. So remember, you always have these two contributions.

And so if you ever have a random variable that you need to take an absolute value of, you don't have to be scared. All you have to do is consider both of these contributions and add them up, and you have the PDF that you want. So I'll see you next time.

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