6.254 : Game Theory with Engineering Applications
Lecture 7: Supermodular Games

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Outline

- Uniqueness of a Pure Nash Equilibrium for Continuous Games
- Supermodular Games

- Reading:
  - Rosen J.B., "Existence and uniqueness of equilibrium points for concave N-person games," *Econometrica*, vol. 33, no. 3, 1965.
  - Fudenberg and Tirole, Section 12.3.

Uniqueness of a Pure Strategy Nash Equilibrium in Continuous Games

- We have shown in the previous lecture the following result:
  - Given a strategic form game $\langle \mathcal{I}, (S_i), (u_i) \rangle$, assume that the strategy sets S_i are nonempty, convex, and compact sets, $u_i(s)$ is continuous in s, and $u_i(s_i, s_{-i})$ is quasiconcave in s_i. Then the game $\langle \mathcal{I}, (S_i), (u_i) \rangle$ has a pure strategy Nash equilibrium.
- We have seen an example that shows that even under strict convexity assumptions, there may be infinitely many pure strategy Nash equilibria.

Uniqueness of a Pure Strategy Nash Equilibrium

- We will next establish conditions that guarantee that a strategic form game has a unique pure strategy Nash equilibrium, following the classical paper [Rosen 65].

Notation:

- Given a scalar-valued function $f : \mathbb{R}^n \mapsto \mathbb{R}$, we use the notation $\nabla f(x)$ to denote the gradient vector of f at point x, i.e.,

$$\nabla f(x) = \left[\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n}\right]^T.$$

- Given a scalar-valued function $u : \prod_{i=1}^{I} \mathbb{R}^{m_i} \mapsto \mathbb{R}$, we use the notation $\nabla_i u(x)$ to denote the gradient vector of u with respect to x_i at point x, i.e.,

$$\nabla_i u(x) = \left[\frac{\partial u(x)}{\partial x_i^1}, \ldots, \frac{\partial u(x)}{\partial x_i^{m_i}}\right]^T. \quad (1)$$

Optimality Conditions for Nonlinear Optimization Problems

Theorem (3)

(Karush-Kuhn-Tucker conditions) *Let x^* be an optimal solution of the optimization problem*

$$\begin{aligned} &\textit{maximize} && f(x) \\ &\textit{subject to} && g_j(x) \geq 0, \qquad j = 1, \ldots, r, \end{aligned}$$

where the cost function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and the constraint functions $g_j : \mathbb{R}^n \mapsto \mathbb{R}$ are continuously differentiable. Denote the set of active constraints at x^* as $A(x^*) = \{j = 1, \ldots, r \mid g_j(x^*) = 0\}$. Assume that the active constraint gradients, $\nabla g_j(x^*)$, $j \in A(x^*)$, are linearly independent vectors. Then, there exists a nonnegative vector $\lambda^* \in \mathbb{R}^r$ (Lagrange multiplier vector) such that

$$\nabla f(x^*) + \sum_{j=1}^{r} \lambda_j^* \nabla g_j(x^*) = 0,$$

$$\lambda_j^* g_j(x^*) = 0, \qquad \forall\, j = 1, \ldots, r. \tag{2}$$

Optimality Conditions for Nonlinear Optimization Problems

For convex optimization problems (i.e., minimizing a convex function (or maximizing a concave function) over a convex constraint set), we can provide necessary and sufficient conditions for the optimality of a feasible solution:

Theorem (4)

Consider the optimization problem

$$\begin{array}{lll} \textit{maximize} & f(x) & \\ \textit{subject to} & g_j(x) \geq 0, & j = 1, \ldots, r, \end{array}$$

where the cost function $f : \mathbb{R}^n \mapsto \mathbb{R}$ *and the constraint functions* $g_j : \mathbb{R}^n \mapsto \mathbb{R}$ *are concave functions. Assume also that there exists some* $\bar{x}$ *such that* $g_j(\bar{x}) > 0$ *for all* $j = 1, \ldots, r$. *Then a vector* $x^* \in \mathbb{R}^n$ *is an optimal solution of the preceding problem if and only if* $g_j(x^*) \geq 0$ *for all* $j = 1, \ldots, r$, *and there exists a nonnegative vector* $\lambda^* \in \mathbb{R}^r$ *(Lagrange multiplier vector) such that*

$$\nabla f(x^*) + \sum_{j=1}^{r} \lambda_j^* \nabla g_j(x^*) = 0,$$

$$\lambda_j^* g_j(x^*) = 0, \qquad \forall\, j = 1, \ldots, r.$$

Uniqueness of a Pure Strategy Nash Equilibrium

- We now return to the analysis of the uniqueness of a pure strategy equilibrium in strategic form games.
- We assume that for player $i \in \mathcal{I}$, the strategy set S_i is given by

$$S_i = \{x_i \in \mathbb{R}^{m_i} \mid h_i(x_i) \geq 0\}, \tag{3}$$

  where $h_i : \mathbb{R}^{m_i} \mapsto \mathbb{R}$ is a concave function.
- Since h_i is concave, it follows that the set S_i is a convex set (exercise!).
- Therefore the set of strategy profiles $S = \prod_{i=1}^{I} S_i \subset \prod_{i=1}^{I} \mathbb{R}^{m_i}$, being a Cartesian product of convex sets, is a convex set.
- Given these strategy sets, a vector $x^* \in \prod_{i=1}^{I} \mathbb{R}^{m_i}$ is a pure strategy Nash equilibrium if and only if for all $i \in \mathcal{I}$, x_i^* is an optimal solution of

$$\begin{aligned} \text{maximize}_{y_i \in \mathbb{R}^{m_i}} \quad & u_i(y_i, x_{-i}^*) \\ \text{subject to} \quad & h_i(y_i) \geq 0. \end{aligned} \tag{4}$$

- We use the notation $\nabla u(x)$ to denote

$$\nabla u(x) = [\nabla_1 u_1(x), \ldots, \nabla_I u_I(x)]^T. \tag{5}$$

Uniqueness of a Pure Strategy Nash Equilibrium

- We introduce the key condition for uniqueness of a pure strategy Nash equilibrium.

Definition

We say that the payoff functions $(u_1, \ldots, u_I)$ *are* **diagonally strictly concave for** $x \in S$, *if for every* $x^*, \bar{x} \in S$, *we have*

$$(\bar{x} - x^*)^T \nabla u(x^*) + (x^* - \bar{x})^T \nabla u(\bar{x}) > 0.$$

Theorem

Consider a strategic form game $\langle \mathcal{I}, (S_i), (u_i) \rangle$. *For all* $i \in \mathcal{I}$, *assume that the strategy sets* S_i *are given by Eq. (3), where* h_i *is a concave function, and there exists some* $\tilde{x}_i \in \mathbb{R}^{m_i}$ *such that* $h_i(\tilde{x}_i) > 0$. *Assume also that the payoff functions* $(u_1, \ldots, u_I)$ *are diagonally strictly concave for* $x \in S$. *Then the game has a unique pure strategy Nash equilibrium.*

Proof

- Assume that there are two distinct pure strategy Nash equilibria.
- Since for each $i \in \mathcal{I}$, both x_i^* and $\bar{x}_i$ must be an optimal solution for an optimization problem of the form (4), Theorem 2 implies the existence of nonnegative vectors $\lambda^* = [\lambda_1^*, \ldots, \lambda_I^*]^T$ and $\bar{\lambda} = [\bar{\lambda}_1, \ldots, \bar{\lambda}_I]^T$ such that for all $i \in \mathcal{I}$, we have

$$\nabla_i u_i(x^*) + \lambda_i^* \nabla h_i(x_i^*) = 0, \tag{6}$$

$$\lambda_i^* h_i(x_i^*) = 0, \tag{7}$$

and

$$\nabla_i u_i(\bar{x}) + \bar{\lambda}_i \nabla h_i(\bar{x}_i) = 0, \tag{8}$$

$$\bar{\lambda}_i h_i(\bar{x}_i) = 0. \tag{9}$$

Proof

- Multiplying Eqs. (6) and (8) by $(\bar{x}_i - x_i^*)^T$ and $(x_i^* - \bar{x}_i)^T$ respectively, and adding over all $i \in \mathcal{I}$, we obtain

$$
\begin{aligned}
0 &= (\bar{x} - x^*)^T \nabla u(x^*) + (x^* - \bar{x})^T \nabla u(\bar{x}) \qquad (10) \\
&\quad + \sum_{i \in \mathcal{I}} \lambda_i^* \nabla h_i(x_i^*)^T (\bar{x}_i - x_i^*) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla h_i(\bar{x}_i)^T (x_i^* - \bar{x}_i) \\
&\quad > \sum_{i \in \mathcal{I}} \lambda_i^* \nabla h_i(x_i^*)^T (\bar{x}_i - x_i^*) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla h_i(\bar{x}_i)^T (x_i^* - \bar{x}_i),
\end{aligned}
$$

  where to get the strict inequality, we used the assumption that the payoff functions are diagonally strictly concave for $x \in S$.

- Since the h_i are concave functions, we have

$$
h_i(x_i^*) + \nabla h_i(x_i^*)^T (\bar{x}_i - x_i^*) \geq h_i(\bar{x}_i).
$$

Proof

- Using the preceding together with $\lambda_i^* > 0$, we obtain for all i,

$$\begin{aligned}\lambda_i^* \nabla h_i(x_i^*)^T (\bar{x}_i - x_i^*) &\geq \lambda_i^* (h_i(\bar{x}_i) - h_i(x_i^*)) \\ &= \lambda_i^* h_i(\bar{x}_i) \\ &\geq 0,\end{aligned}$$

  where to get the equality we used Eq. (7), and to get the last inequality, we used the facts $\lambda_i^* > 0$ and $h_i(\bar{x}_i) \geq 0$.

- Similarly, we have

$$\bar{\lambda}_i \nabla h_i(\bar{x}_i)^T (x_i^* - \bar{x}_i) \geq 0.$$

- Combining the preceding two relations with the relation in (10) yields a contradiction, thus concluding the proof.

Sufficient Condition for Diagonal Strict Concavity

- Let $U(x)$ denote the Jacobian of $\nabla u(x)$ [see Eq. (5)]. In particular, if the x_i are all 1-dimensional, then $U(x)$ is given by

$$U(x) = \begin{pmatrix} \frac{\partial^2 u_1(x)}{\partial x_1^2} & \frac{\partial^2 u_1(x)}{\partial x_1 \partial x_2} & \cdots \\ \frac{\partial^2 u_2(x)}{\partial x_2 \partial x_1} & \ddots & \\ \vdots & & \end{pmatrix}.$$

Proposition

For all $i \in \mathcal{I}$, assume that the strategy sets S_i are given by Eq. (3), where h_i is a concave function. Assume that the symmetric matrix $(U(x) + U^T(x))$ is negative definite for all $x \in S$, i.e., for all $x \in S$, we have

$$y^T(U(x) + U^T(x))y < 0, \qquad \forall\ y \neq 0.$$

Then, the payoff functions $(u_1, \ldots, u_I)$ are diagonally strictly concave for $x \in S$.

Proof

- Let $x^*,\ \bar{x} \in S$. Consider the vector

$$x(\lambda) = \lambda x^* + (1-\lambda)\bar{x}, \qquad \text{for some } \lambda \in [0,1].$$

  Since S is a convex set, $x(\lambda) \in S$.

- Because $U(x)$ is the Jacobian of $\nabla u(x)$, we have

$$\begin{aligned} \frac{d}{d\lambda}\nabla u(x(\lambda)) &= U(x(\lambda))\frac{dx(\lambda)}{d(\lambda)} \\ &= U(x(\lambda))(x^* - \bar{x}), \end{aligned}$$

  or

$$\int_0^1 U(x(\lambda))(x^* - \bar{x})d\lambda = \nabla u(x^*) - \nabla u(\bar{x}).$$

Proof

- Multiplying the preceding by $(\bar{x} - x^*)^T$ yields

$$\begin{aligned}(\bar{x} - x^*)^T \nabla u(x^*) \quad &+ \quad (x^* - \bar{x})^T \nabla u(\bar{x}) \\ &= -\frac{1}{2} \int_0^1 (x^* - \bar{x})^T [U(x(\lambda)) + U^T(x(\lambda))](x^* - \bar{x}) d\lambda \\ &> 0,\end{aligned}$$

where to get the strict inequality we used the assumption that the symmetric matrix $(U(x) + U^T(x))$ is negative definite for all $x \in S$.

Supermodular Games

- Supermodular games are those characterized by strategic complementarities
- Informally, this means that the marginal utility of increasing a player's strategy raises with increases in the other players' strategies.
    - Implication $\Rightarrow$ best response of a player is a nondecreasing function of other players' strategies
- Why interesting?
    - They arise in many models.
    - Existence of a pure strategy equilibrium without requiring the quasi-concavity of the payoff functions.
    - Many solution concepts yield the same predictions.
    - The equilibrium set has a smallest and a largest element.
    - They have nice sensitivity (or comparative statics) properties and behave well under a variety of distributed dynamic rules.
- Much of the theory is due to [Topkis 79, 98], [Milgrom and Roberts 90], [Milgrom and Shannon 94], and [Vives 90, 01].

Lattices and Tarski's Theorem

- The machinery needed to study supermodular games is lattice theory and monotonicity results in lattice programming.
  - Methods used are non-topological and they exploit order properties
- We first briefly summarize some preliminaries related to lattices.

Definition

- *Given a set S and a binary relation $\geq$, the pair $(S, \geq)$ is a* **partially ordered set** *if $\geq$ is reflexive ($x \geq x$ for all $x \in S$), transitive ($x \geq y$ and $y \geq z$ implies that $x \geq z$), and antisymmetric ($x \geq y$ and $y \geq x$ implies that $x = y$).*
- *A partially ordered set $(S, \geq)$ is* **(completely) ordered** *if for $x \in S$ and $y \in S$, either $x \geq y$ or $y \geq x$.*

Lattices

Definition

A lattice is a partially ordered set $(S, \geq)$ *s.t. any two elements* x, y *have a least upper bound (supremum),* $\sup_S(x, y) = \inf\{z \in S | z \geq x,\ z \geq y\}$, *and a greatest lower bound (infimum),* $\inf_S(x, y) = \sup\{z \in S | z \leq x,\ z \leq y\}$ *in the set.*

- Supremum of $\{x, y\}$ is denoted by $x \vee y$ and is called the join of x and y.
- Infimum of $\{x, y\}$ is denoted by $x \wedge y$ and is called the meet of x and y.

Examples:

- Any interval of the real line with the usual order is a lattice, since any two points have a supremum and infimum in the interval.
- However, the set $S \subset \mathbb{R}^2$, $S = \{(1, 0), (0, 1)\}$, is not a lattice with the vector ordering (the usual componentwise ordering: $x \leq y$ if and only if $x_i \leq y_i$ for any i), since $(1, 0)$ and $(0, 1)$ have no joint upper bound in S.
- $S' = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ is a lattice with the vector ordering.
- Similarly, the simplex in $\mathbb{R}^n$ (again with the usual vector ordering) $\{x \in \mathbb{R}^n \mid \sum_i x_i = 1,\ x_i \geq 0\}$ is not a lattice, while the box $\{x \in \mathbb{R}^n \mid 0 \leq x_1 \leq 1\}$ is.

Lattices

Definition

A lattice $(S, \geq)$ *is* **complete** *if every nonempty subset of* S *has a supremum and an infimum in* S.

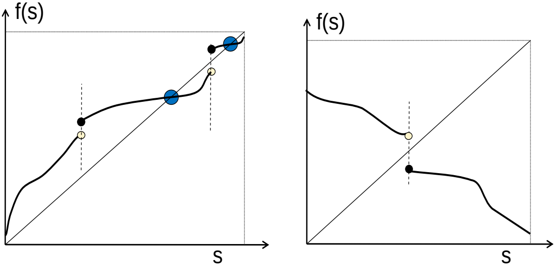
- Any compact interval of the real line with the usual order is a complete lattice, while the open interval (a, b) is a lattice but is not complete [indeed the supremum of (a, b) does not belong to (a, b)].

Tarski's Fixed Point Theorem

- We state the lattice theoretical fixed point theorem due to Tarski.
- Let $(S, \geq)$ be a partially ordered set. A function f from S to S is increasing if for all $x, y \in S$, $x \geq y$ implies $f(x) \geq f(y)$.

Theorem (Tarski)

Let $(S, \geq)$ be a complete lattice and $f : S \to S$ an increasing function. Then, the set of fixed points of f, denoted by E, is nonempty and $(E, \geq)$ is a complete lattice.



Supermodularity of a Function

Definition

Let $(X, \geq)$ be a lattice. A function $f : X \to \mathbb{R}$ is supermodular on S if for all $x, y \in X$

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y).$$

Note that supermodularity is automatically satisfied if X is single dimensional.

Monotonicity of Optimal Solutions

- From now on, we will assume that $X \subseteq \mathbb{R}$.
  - The following analysis and theory extends to the case where X is a lattice.
- We first study the monotonicity properties of optimal solutions of parametric optimization problems. Consider a problem

$$x(t) = \arg\max_{x \in X} f(x,t),$$

  where $f : X \times T \to \mathbb{R}$, $X \subseteq \mathbb{R}$, and T is some partially ordered set.
- We will mostly focus on $T \subseteq \mathbb{R}^K$ with the usual **vector order**, i.e., for some $x, y \in T$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, \ldots, k$.
- We are interested in conditions under which we can establish that $x(t)$ is a nondecreasing function of t.

Increasing Differences

- Key property: Increasing differences.

Definition

Let $X \subseteq \mathbb{R}$ and T be some partially ordered set. A function $f : X \times T \to \mathbb{R}$ has **increasing differences** *in (x, t) if for all $x' \geq x$ and $t' \geq t$, we have*

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t).$$

- **Intuitively**: incremental gain to choosing a higher x (i.e., x' rather than x) is greater when t is higher, i.e., $f(x', t) - f(x, t)$ is nondecreasing in t.
- You can check that the property of increasing differences is symmetric : an equivalent statement is that if $t' > t$, then $f(x, t') - f(x, t)$ is nondecreasing in x.
- The previous definition gives an abstract characterization. The following result makes checking increasing differences easy in many cases.

Increasing Differences

Lemma

Let $X \subset \mathbb{R}$ and $T \subset \mathbb{R}^k$ for some k, a partially ordered set with the usual vector order. Let $f : X \times T \to \mathbb{R}$ be a twice continuously differentiable function. Then, the following statements are equivalent:

- *The function f has increasing differences in (x, t).*
- *For all $t' \geq t$ and all $x \in X$, we have*

$$\frac{\partial f(x, t')}{\partial x} \geq \frac{\partial f(x, t)}{\partial x}.$$

- *For all $x \in X$, $t \in T$, and all $i = 1, \ldots, k$, we have*

$$\frac{\partial^2 f(x, t)}{\partial x \partial t_i} \geq 0.$$

Example I – Network effects (positive externalities)

- A set $\mathcal{I}$ of users can use one of two products X and Y (e.g., Blu-ray and HD DVD).
- $B_i(J, k)$ denotes payoff to i when a subset J of users use k and $i \in J$.
- There exists a **positive externality** if

$$B_i(J, k) \leq B_i(J', k), \qquad \text{when } J \subset J',$$

  i.e., player i better off if more users use the same technology as him.
- This leads to a strategic form game with actions $S_i = \{X, Y\}$
- Define the order $Y \succeq X$, which induces a lattice structure
- Given $s \in S$, let $X(s) = \{i \in \mathcal{I} \mid s_i = X\}$, $Y(s) = \{i \in \mathcal{I} \mid s_i = Y\}$.
- We define the payoff functions as

$$u_i(s_i, s_{-i}) = \begin{cases} B_i(X(s), X) & \text{if } s_i = X, \\ B_i(Y(s), Y) & \text{if } s_i = Y \end{cases}$$

- It can be verified that payoff functions satisfy increasing differences.

Example II– Cournot As a Supermodular Game with Change of Order

- Consider Cournot duopoly model. Two firms choose the quantity they produce $q_i \in [0, \infty)$.
- Let $P(Q)$ with $Q = q_i + q_j$ denote the inverse demand (price) function. Payoff function of each firm is $u_i(q_i, q_j) = q_i P(q_i + q_j) - cq_i$.
- Assume $P'(Q) + q_i P''(Q) \leq 0$ (firm i's marginal revenue decreasing in q_j).
- We can now verify that the payoff functions of the transformed game defined by $s_1 = q_1$, $s_2 = -q_2$ have increasing differences in (s_1, s_2).

Monotonicity of Optimal Solutions

- Key theorem about monotonicity of optimal solutions:

Theorem (Topkis)

Let $X \subset \mathbb{R}$ be a compact set and T be some partially ordered set. Assume that the function $f : X \times T \to \mathbb{R}$ is continuous [or upper semicontinuous] in x for all $t \in T$ and has increasing differences in (x, t). Define $x(t) \equiv \arg\max_{x \in X} f(x, t)$. Then, we have:

1. *For all $t \in T$, $x(t)$ is nonempty and has a greatest and least element, denoted by $\bar{x}(t)$ and $\underline{x}(t)$ respectively.*
2. *For all $t' \geq t$, we have $\bar{x}(t') \geq \bar{x}(t)$ and $\underline{x}(t') \geq \underline{x}(t)$.*

- Summary: if f has increasing differences, the set of optimal solutions $x(t)$ is non-decreasing in the sense that the largest and the smallest selections are non-decreasing.

Proof

- By the assumptions that for all $t \in T$, the function $f(\cdot, t)$ is upper semicontinuous and X is compact, it follows by the Weierstrass' Theorem that $x(t)$ is nonempty. For all $t \in T$, $x(t) \subset X$, therefore is bounded.
- Since $X \subset \mathbb{R}$, to establish that $x(t)$ has a greatest and lowest element, it suffices to show that $x(t)$ is closed.
- Let $\{x^k\}$ be a sequence in $x(t)$. Since X is compact, x^k has a limit point $\bar{x}$. By restricting to a subsequence if necessary, we may assume without loss of generality that x^k converges to $\bar{x}$.
- Since $x^k \in x(t)$ for all k, we have

$$f(x^k, t) \geq f(x, t), \qquad \forall\, x \in X.$$

Taking the limit as $k \to \infty$ in the preceding relation and using the upper semicontinuity of $f(\cdot, t)$, we obtain

$$f(\bar{x}, t) \geq \limsup_{k \to \infty} f(x^k, t) \geq f(x, t), \qquad \forall\, x \in X,$$

thus showing that $\bar{x}$ belongs to $x(t)$, and proving the closedness claim.

Proof

- Let $t' \geq t$. Let $x \in x(t)$ and $x' = \bar{x}(t')$.
- By the fact that x maximizes $f(x, t)$, we have

$$f(x, t) - f(\min(x, x'), t) \geq 0.$$

- This implies (check the two cases: $x \geq x'$ and $x' \geq x$) that

$$f(\max(x, x'), t) - f(x', t) \geq 0.$$

- By increasing differences of f, this yields

$$f(\max(x, x'), t') - f(x', t') \geq 0.$$

- Thus $\max(x, x')$ maximizes $f(\cdot, t')$, i.e, $\max(x, x')$ belongs to $x(t')$. Since x' is the greatest element of the set $x(t')$, we conclude that $\max(x, x') \leq x'$, thus $x \leq x'$.
- Since x is an arbitrary element of $x(t)$, this implies $\bar{x}(t) \leq \bar{x}(t')$. A similar argument applies to the smallest maximizers.

Supermodular Games

Definition

The strategic game $\langle \mathcal{I}, (S_i), (u_i) \rangle$ is a supermodular game if for all $i \in \mathcal{I}$:

- *S_i is a compact subset of $\mathbb{R}$ [or more generally S_i is a complete lattice in $\mathbb{R}^{m_i}$];*
- *u_i is upper semicontinuous in s_i, continuous in s_{-i}.*
- *u_i has increasing differences in (s_i, s_{-i}) [or more generally u_i is supermodular in (s_i, s_{-i}), which is an extension of the property of increasing differences to games with multi-dimensional strategy spaces].*

Supermodular Games

- Applying Topkis' theorem implies that each player's "best response correspondence is increasing in the actions of other players".

Corollary

Assume $\langle \mathcal{I}, (S_i), (u_i) \rangle$ *is a supermodular game. Let*

$$B_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

Then:

1. $B_i(s_{-i})$ *has a greatest and least element, denoted by* $\bar{B}_i(s_{-i})$ *and* $\underline{B}_i(s_{-i})$.
2. *If* $s'_{-i} \geq s_{-i}$*, then* $\bar{B}_i(s'_{-i}) \geq \bar{B}_i(s_{-i})$ *and* $\underline{B}_i(s'_{-i}) \geq \underline{B}_i(s_{-i})$.

- Applying Tarski's fixed point theorem to $\bar{B}$ establishes the existence of a pure Nash equilibrium for any supermodular game.
- We next pursue a different approach which provides more insight into the structure of Nash equilibria.

Supermodular Games

Theorem (Milgrom and Roberts)

Let $\langle \mathcal{I}, (S_i), (u_i) \rangle$ be a supermodular game. Then the set of strategies that survive iterated strict dominance in pure strategies has greatest and least elements $\bar{s}$ and $\underline{s}$, coinciding with the greatest and the least pure strategy Nash Equilibria.

Corollary

Supermodular games have the following properties:

1. *Pure strategy NE exist.*
2. *The largest and smallest strategies are compatible with iterated strict dominance (ISD), rationalizability, correlated equilibrium, and Nash equilibrium are the same.*
3. *If a supermodular game has a unique NE, it is dominance solvable (and lots of learning and adjustment rules converge to it, e.g., best-response dynamics).*

Proof

- We iterate the best response mapping. Let $S^0 = S$, and let $s^0 = (s_1^0, \ldots, s_I^0)$ be the largest element of S.
- Let $s_i^1 = \bar{B}_i(s_{-i}^0)$ and $S_i^1 = \{s_i \in S_i^0 \mid s_i \leq s_i^1\}$.
- We show that any $s_i > s_i^1$, i.e, any $s_i \notin S_i^1$, is strictly dominated by s_i^1. For all $s_{-i} \in S_{-i}$, we have

$$\begin{aligned} u_i(s_i, s_{-i}) - u_i(s_i^1, s_{-i}) &\leq u_i(s_i, s_{-i}^0) - u_i(s_i^1, s_{-i}^0) \\ &< 0, \end{aligned}$$

  where the first inequality follows by the increasing differences of $u_i(s_i, s_{-i})$ in (s_i, s_{-i}), and the strict inequality follows by the fact that s_i is not a best response to s_{-i}^0.
- Note that $s_i^1 \leq s_i^0$.
- Iterating this argument, we define

$$s_i^k = \bar{B}_i(s_{-i}^{k-1}), \qquad S_i^k = \{s_i \in S_i^{k-1} \mid s_i \leq s_i^k\}.$$

Proof

- Assume $s^k \leq s^{k-1}$. Then, by Corollary (Topkis), we have

$$s_i^{k+1} = \bar{B}_i(s_{-i}^k) \leq \bar{B}_i(s_{-i}^{k-1}) = s_i^k.$$

- This shows that the sequence $\{s_i^k\}$ is a decreasing sequence, which is bounded from below, and hence it has a limit, which we denote by $\bar{s}_i$. Only the strategies $s_i \leq \bar{s}_i$ are undominated. Similarly, we can start with $s^0 = (s_1^0, \ldots, s_I^0)$ the smallest element in S and identify $\underline{s}$.
- To complete the proof, we show that $\bar{s}$ and $\underline{s}$ are NE. By construction, for all i and $s_i \in S_i$, we have

$$u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i, s_{-i}^k).$$

- Taking the limit as $k \to \infty$ in the preceding relation and using the upper semicontinuity of u_i in s_i and continuity of u_i in s_{-i}, we obtain

$$u_i(\bar{s}_i, \bar{s}_{-i}) \geq u_i(s_i, \bar{s}_{-i}),$$

showing the desired claim.

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