6.254 : Game Theory with Engineering Applications
Lecture 8: Supermodular and Potential Games

Asu Ozdaglar
MIT

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Outline

- Review of Supermodular Games
- Potential Games

- Reading:
  - Fudenberg and Tirole, Section 12.3.
  - Monderer and Shapley, "Potential Games," Games and Economic Behavior, vol. 14, pp. 124-143, 1996.

Supermodular Games

- Supermodular games are those characterized by strategic complementarities
- Informally, this means that the marginal utility of increasing a player's strategy raises with increases in the other players' strategies.
  - Implication $\Rightarrow$ best response of a player is a nondecreasing function of other players' strategies
- Why interesting?
  - They arise in many models.
  - Existence of a pure strategy equilibrium without requiring the quasi-concavity of the payoff functions.
  - Many solution concepts yield the same predictions.
  - The equilibrium set has a smallest and a largest element.
  - They have nice sensitivity (or comparative statics) properties and behave well under a variety of distributed dynamic rules.
- Much of the theory is due to [Topkis 79, 98], [Milgrom and Roberts 90], [Milgrom and Shannon 94], and [Vives 90, 01].

Increasing Differences

- Key property: Increasing differences.

Definition

Let $X \subseteq \mathbb{R}$ and T be some partially ordered set. A function $f : X \times T \to \mathbb{R}$ has **increasing differences** *in (x, t) if for all $x' \geq x$ and $t' \geq t$, we have*

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t).$$

- **Intuitively**: incremental gain to choosing a higher x (i.e., x' rather than x) is greater when t is higher, i.e., $f(x', t) - f(x, t)$ is nondecreasing in t.
- You can check that the property of increasing differences is symmetric : an equivalent statement is that if $t' \geq t$, then $f(x, t') - f(x, t)$ is nondecreasing in x.
- The previous definition gives an abstract characterization. The following result makes checking increasing differences easy in many cases.

Increasing Differences

Lemma

Let $X \subset \mathbb{R}$ and $T \subset \mathbb{R}^k$ for some k, a partially ordered set with the usual vector order. Let $f : X \times T \to \mathbb{R}$ be a twice continuously differentiable function. Then, the following statements are equivalent:

- *The function f has increasing differences in (x, t).*
- *For all $t' \geq t$ and all $x \in X$, we have*

$$\frac{\partial f(x, t')}{\partial x} \geq \frac{\partial f(x, t)}{\partial x}.$$

- *For all $x \in X$, $t \in T$, and all $i = 1, \ldots, k$, we have*

$$\frac{\partial^2 f(x, t)}{\partial x \partial t_i} \geq 0.$$

Monotonicity of Optimal Solutions

- Key theorem about monotonicity of optimal solutions:

Theorem (Topkis)

Let $X \subset \mathbb{R}$ be a compact set and T be some partially ordered set. Assume that the function $f : X \times T \to \mathbb{R}$ is continuous [or upper semicontinuous] in x for all $t \in T$ and has increasing differences in (x, t). Define $x(t) \equiv \arg\max_{x \in X} f(x, t)$. Then, we have:

1. *For all $t \in T$, $x(t)$ is nonempty and has a greatest and least element, denoted by $\bar{x}(t)$ and $\underline{x}(t)$ respectively.*
2. *For all $t' \geq t$, we have $\bar{x}(t') \geq \bar{x}(t)$ and $\underline{x}(t') \geq \underline{x}(t)$.*

- Summary: if f has increasing differences, the set of optimal solutions $x(t)$ is non-decreasing in the sense that the largest and the smallest selections are non-decreasing.

Supermodular Games

Definition

The strategic game $\langle \mathcal{I}, (S_i), (u_i) \rangle$ is a supermodular game if for all $i \in \mathcal{I}$:

- *S_i is a compact subset of $\mathbb{R}$ [or more generally S_i is a complete lattice in $\mathbb{R}^{m_i}$];*
- *u_i is upper semicontinuous in s_i, continuous in s_{-i}.*
- *u_i has increasing differences in (s_i, s_{-i}) [or more generally u_i is supermodular in (s_i, s_{-i}), which is an extension of the property of increasing differences to games with multi-dimensional strategy spaces].*

Supermodular Games

- Applying Topkis' theorem implies that each player's "best response correspondence is increasing in the actions of other players".

Corollary

Assume $\langle \mathcal{I}, (S_i), (u_i) \rangle$ is a supermodular game. Let

$$B_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

Then:

1. *$B_i(s_{-i})$ has a greatest and least element, denoted by $\bar{B}_i(s_{-i})$ and $\underline{B}_i(s_{-i})$.*
2. *If $s'_{-i} \geq s_{-i}$, then $\bar{B}_i(s'_{-i}) \geq \bar{B}_i(s_{-i})$ and $\underline{B}_i(s'_{-i}) \geq \underline{B}_i(s_{-i})$.*

- Applying Tarski's fixed point theorem to $\bar{B}$ establishes the existence of a pure Nash equilibrium for any supermodular game.
- We next pursue a different approach which provides more insight into the structure of Nash equilibria.

Supermodular Games

Theorem (Milgrom and Roberts)

Let $\langle \mathcal{I}, (S_i), (u_i) \rangle$ be a supermodular game. Then the set of strategies that survive iterated strict dominance in pure strategies has greatest and least elements $\bar{s}$ and $\underline{s}$, coinciding with the greatest and the least pure strategy Nash Equilibria.

Corollary

Supermodular games have the following properties:

1. *Pure strategy NE exist.*
2. *The largest and smallest strategies are compatible with iterated strict dominance (ISD), rationalizability, correlated equilibrium, and Nash equilibrium are the same.*
3. *If a supermodular game has a unique NE, it is dominance solvable (and lots of learning and adjustment rules converge to it, e.g., best-response dynamics).*

Proof

- We iterate the best response mapping. Let $S^0 = S$, and let $s^0 = (s_1^0, \ldots, s_I^0)$ be the largest element of S.
- Let $s_i^1 = \bar{B}_i(s_{-i}^0)$ and $S_i^1 = \{s_i \in S_i^0 \mid s_i \leq s_i^1\}$.
- We show that any $s_i > s_i^1$, i.e, any $s_i \notin S_i^1$, is strictly dominated by s_i^1. For all $s_{-i} \in S_{-i}$, we have

$$\begin{aligned} u_i(s_i, s_{-i}) - u_i(s_i^1, s_{-i}) &\leq u_i(s_i, s_{-i}^0) - u_i(s_i^1, s_{-i}^0) \\ &< 0, \end{aligned}$$

  where the first inequality follows by the increasing differences of $u_i(s_i, s_{-i})$ in (s_i, s_{-i}), and the strict inequality follows by the fact that s_i is not a best response to s_{-i}^0.
- Note that $s_i^1 \leq s_i^0$.
- Iterating this argument, we define

$$s_i^k = \bar{B}_i(s_{-i}^{k-1}), \qquad S_i^k = \{s_i \in S_i^{k-1} \mid s_i \leq s_i^k\}.$$

Proof

- Assume $s^k \leq s^{k-1}$. Then, by Corollary (Topkis), we have

$$s_i^{k+1} = \bar{B}_i(s_{-i}^k) \leq \bar{B}_i(s_{-i}^{k-1}) = s_i^k.$$

- This shows that the sequence $\{s_i^k\}$ is a decreasing sequence, which is bounded from below, and hence it has a limit, which we denote by $\bar{s}_i$. Only the strategies $s_i \leq \bar{s}_i$ are undominated. Similarly, we can start with $s^0 = (s_1^0, \ldots, s_I^0)$ the smallest element in S and identify $\underline{s}$.
- To complete the proof, we show that $\bar{s}$ and $\underline{s}$ are NE. By construction, for all i and $s_i \in S_i$, we have

$$u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i, s_{-i}^k).$$

- Taking the limit as $k \to \infty$ in the preceding relation and using the upper semicontinuity of u_i in s_i and continuity of u_i in s_{-i}, we obtain

$$u_i(\bar{s}_i, \bar{s}_{-i}) \geq u_i(s_i, \bar{s}_{-i}),$$

showing the desired claim.

Potential Games

- A strategic form game is a **potential game** [ordinal potential game, exact potential game] if there exists a function $\Phi : S \to \mathbb{R}$ such that $\Phi(s_i, s_{-i})$ gives information about $u_i(s_i, s_{-i})$ for each $i \in \mathcal{I}$.
- If so, Φ is referred to as the **potential function.**
- The potential function has a natural analogy to "energy" in physical systems. It will be useful both for locating pure strategy Nash equilibria and also for the analysis of "myopic" dynamics.

Potential Functions and Games

Let $G = \langle \mathcal{I}, (S_i), (u_i) \rangle$ be a strategic form game.

Definition

A function $\Phi : S \to \mathbb{R}$ is called an ordinal potential function for the game G if for each $i \in \mathcal{I}$ and all $s_{-i} \in S_{-i}$,

$$u_i(x, s_{-i}) - u_i(z, s_{-i}) > 0 \text{ iff } \Phi(x, s_{-i}) - \Phi(z, s_{-i}) > 0, \text{ for all } x, z \in S_i.$$

G is called an ordinal potential game if it admits an ordinal potential.

Definition

A function $\Phi : S \to \mathbb{R}$ is called an (exact) potential function for the game G if for each $i \in \mathcal{I}$ and all $s_{-i} \in S_{-i}$,

$$u_i(x, s_{-i}) - u_i(z, s_{-i}) = \Phi(x, s_{-i}) - \Phi(z, s_{-i}), \text{ for all } x, z \in S_i.$$

G is called an (exact) potential game if it admits a potential.

Example

- A potential function assigns a real value for every $s \in S$.
- Thus, when we represent the game payoffs with a matrix (in finite games), we can also represent the potential function as a matrix, each entry corresponding to the vector of strategies from the payoff matrix.

Example

The matrix P is a potential for the "Prisoner's dilemma" game described below:

$$G = \begin{pmatrix} (1,1) & (9,0) \\ (0,9) & (6,6) \end{pmatrix}, \qquad P = \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}$$

Pure Strategy Nash Equilibria in Ordinal Potential Games

Theorem

Every finite ordinal potential game has at least one pure strategy Nash equilibrium.

- **Proof:** The global maximum of an ordinal potential function is a pure strategy Nash equilibrium. To see this, suppose that s^* corresponds to the global maximum. Then, for any $i \in \mathcal{I}$, we have, by definition, $\Phi(s_i^*, s_{-i}^*) - \Phi(s, s_{-i}^*) \geq 0$ for all $s \in S_i$. But since Φ is a potential function, for all i and all $s \in S_i$,

$$u_i(s_i^*, s_{-i}^*) - u_i(s, s_{-i}^*) \geq 0 \quad \text{iff} \quad \Phi(s_i^*, s_{-i}^*) - \Phi(s, s_{-i}^*) \geq 0.$$

  Therefore, $u_i(s_i^*, s_{-i}^*) - u_i(s, s_{-i}^*) \geq 0$ for all $s \in S_i$ and for all $i \in \mathcal{I}$. Hence s^* is a pure strategy Nash equilibrium.
- Note, however, that there may also be other pure strategy Nash equilibria corresponding to local maxima.

Examples of Ordinal Potential Games

- Example: Cournot competition.
- I firms choose quantity $q_i \in (0, \infty)$
- The payoff function for player i given by $u_i(q_i, q_{-i}) = q_i(P(Q) - c)$.
- We define the function $\Phi(q_1, \cdots, q_I) = \left(\prod_{i=1}^{I} q_i\right)(P(Q) - c)$.
- Note that for all i and all $q_{-i} > 0$,

$$u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) > 0 \text{ iff } \Phi(q_i, q_{-i}) - \Phi(q_i', q_{-i}) > 0, \ \forall\ q_i, q_i' > 0.$$

- Φ is therefore an ordinal potential function for this game.

Examples of Exact Potential Games

- Example: Cournot competition (again).
- Suppose now that $P(Q) = a - bQ$ and costs $c_i(q_i)$ are arbitrary.
- We define the function

$$\Phi^*(q_1, \cdots, q_n) = a \sum_{i=1}^{I} q_i - b \sum_{i=1}^{I} q_i^2 - b \sum_{1 \leq i < l \leq I} q_i q_l - \sum_{i=1}^{I} c_i(q_i).$$

- It can be shown that for all i and all q_{-i},

$$u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) = \Phi^*(q_i, q_{-i}) - \Phi^*(q_i', q_{-i}), \text{ for all } q_i, q_i' > 0.$$

- Φ is an exact potential function for this game.

Simple Dynamics in Finite Ordinal Potential Games

Definition

A path in strategy space S is a sequence of strategy vectors $(s^0, s^1, \cdots)$ such that every two consecutive strategies differ in one coordinate (i.e., exactly in one player's strategy).
An improvement path is a path $(s^0, s^1, \cdots)$ such that, $u_{i_k}(s^k) < u_{i_k}(s^{k+1})$ where s^k and s^{k+1} differ in the i_k^{th} coordinate. In other words, the payoff improves for the player who changes his strategy.

- An improvement path can be thought of as generated dynamically by "myopic players", who update their strategies according to **1-sided better reply dynamic**.

Simple Dynamics in Finite Ordinal Potential Games

Proposition

In every finite ordinal potential game, every improvement path is finite.

Proof: Suppose $(s^0, s^1, \cdots)$ is an improvement path. Therefore we have,

$$\Phi(s^0) < \Phi(s^1) < \cdots ,$$

where Φ is the ordinal potential. Since the game is finite, i.e., it has a finite strategy space, the potential function takes on finitely many values and the above sequence must end in finitely many steps.

- This implies that in finite ordinal potential games, every "maximal" improvement path must terminate in an equilibrium point.
- That is, the simple myopic learning process based on 1-sided better reply dynamic converges to the equilibrium set.
- Next week, we will show that other natural simple dynamics also converge to a pure equilibrium for potential games.

Characterization of Finite Exact Potential Games

- For a finite path $\gamma = (s^0, \ldots, s^N)$, let

$$I(\gamma) = \sum_{i=1}^{N} u^{m_i}(s^i) - u^{m_i}(s^{i-1}),$$

where m_i denotes the player changing its strategy in the ith step of the path.
- The path $\gamma = (s^0, \ldots, s^N)$ is closed if $s^0 = s^N$. It is a simple closed path if in addition $s^l \neq s^k$ for every $0 \leq l \neq k \leq N - 1$.

Theorem

A game G is an exact potential game if and only if for all finite simple closed paths, γ, $I(\gamma) = 0$. Moreover, it is sufficient to check simple closed paths of length 4.

Intuition: Let $I(\gamma) \neq 0$. If potential existed then it would increase when the cycle is completed.

Infinite Potential Games

Proposition

Let G be a continuous potential game with compact strategy sets. Then G has at least one pure strategy Nash equilibrium.

Proposition

Let G be a game such that $S_i \subseteq \mathbb{R}$ and the payoff functions $u_i : S \to \mathbb{R}$ are continuously differentiable. Let $\Phi : S \to \mathbb{R}$ be a function. Then, Φ is a potential for G if and only if Φ is continuously differentiable and

$$\frac{\partial u_i(s)}{\partial s_i} = \frac{\partial \Phi(s)}{\partial s_i} \qquad \text{for all } i \in \mathcal{I} \text{ and all } s \in S.$$

Congestion Games

Congestion Model: $C = \langle \mathcal{I}, \mathcal{M}, (S_i)_{i \in \mathcal{I}}, (c^j)_{j \in \mathcal{M}} \rangle$ where:

- $\mathcal{I} = \{1, 2, \cdots, I\}$ is the set of players.
- $\mathcal{M} = \{1, 2, \cdots, m\}$ is the set of resources.
- S_i is the set of resource combinations (e.g., links or common resources) that player i can take/use. A strategy for player i is $s_i \in S_i$, corresponding to the subset of resources that this player is using.
- $c^j(k)$ is the benefit for the negative of the cost to each user who uses resource j if k users are using it.
- Define congestion game $\langle \mathcal{I}, (S_i), (u_i) \rangle$ with utilities

$$u_i(s_i, s_{-i}) = \sum_{j \in s_i} c^j(k_j),$$

where k_j is the number of users of resource j under strategy s.

Congestion and Potential Games

Theorem (Rosenthal (73))

Every congestion game is a potential game and thus has a pure strategy Nash equilibrium.

- **Proof**: For each j define $\bar{k}_j^i$ as the usage of resource j excluding player i, i.e.,
$$\bar{k}_j^i = \sum_{i' \neq i} \mathbf{I}\left[j \in s_{i'} \right],$$
where $\mathbf{I}\left[j \in s_{i'} \right]$ is the indicator for the event that $j \in s_{i'}$.
- With this notation, the utility difference of player i from two strategies s_i and s_i' (when others are using the strategy profile s_{-i}) is
$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) - \sum_{j \in s_i'} c^j(\bar{k}_j^i + 1).$$

Proof Continued

- Now consider the function

$$\Phi(s) = \sum_{j \in \bigcup_{i' \in \mathcal{I}} s_{i'}} \left[\sum_{k=1}^{k_j} c^j(k) \right].$$

- We can also write

$$\Phi(s_i, s_{-i}) = \sum_{j \in \bigcup\limits_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k) \right] + \sum_{j \in s_i} c^j(\bar{k}_j^i + 1).$$

Proof Continued

- Therefore:

$$\begin{aligned}
\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) &= \sum_{j \in \bigcup_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k) \right] + \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) \\
&\quad - \sum_{j \in \bigcup_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k) \right] + \sum_{j \in s_i'} c^j(\bar{k}_j^i + 1) \\
&= \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) - \sum_{j \in s_i'} c^j(\bar{k}_j^i + 1) \\
&= u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}).
\end{aligned}$$

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