6.254 : Game Theory with Engineering Applications
Lecture 9: Computation of NE in finite games

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Introduction

- In this lecture, we study various approaches for the computation of mixed Nash equilibrium for finite games.
- Our focus will mainly be on two player finite games (i.e., bimatrix games).
- We will also mention extensions to games with multiple players and continuous strategy spaces at the end.
- The two survey papers [von Stengel 02] and [McKelvey and McLennan 96] provide good references for this topic.

Zero-Sum Finite Games

- We consider a zero-sum game where we have two players. Assume that player 1 has n actions and player 2 has m actions.
- We denote the $n \times m$ payoff matrices of player 1 and 2 by A and B.
- Let x denote the mixed strategy of player 1, i.e., $x \in X$, where

$$X = \{x \mid \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0\},$$

and y denote the mixed strategy of player 2, i.e., $y \in Y$, where

$$Y = \{y \mid \sum_{j=1}^{m} y_j = 1,\ y_j \geq 0\}.$$

- Given a mixed strategy profile (x, y), the payoffs of player 1 and player 2 can be expressed in terms of the payoff matrices as,

$$u_1(x, y) = x^T A y,$$

$$u_2(x, y) = x^T B y.$$

Zero-Sum Finite Games

- Recall the definition of a Nash equilibrium: A mixed strategy profile (x^*, y^*) is a mixed strategy Nash equilibrium if and only if

$$(x^*)^T A y^* \geq x^T A y^*, \qquad \text{for all } x \in X,$$

$$(x^*)^T B y^* \geq (x^*)^T B y, \qquad \text{for all } y \in Y.$$

- For zero-sum games, we have $B = -A$, hence the preceding relation becomes

$$(x^*)^T A y^* \leq (x^*)^T A y, \qquad \text{for all } y \in Y.$$

- Combining the preceding, we obtain

$$x^T A y^* \leq (x^*)^T A y^* \leq (x^*)^T A Y, \qquad \text{for all } x \in X, \; y \in Y,$$

i.e., (x^*, y^*) is a saddle point of the function $x^T A y$ defined over $X \times Y$.

- Note that a vector (x^*, y^*) is a saddle point if $x^* \in X$, $y^* \in Y$, and

$$\sup_{x \in X} x^T A y^* = (x^*)^T A y^* = \inf_{y \in Y} (x^*)^T A y. \qquad (1)$$

Zero-Sum Finite Games

- For any function $\phi : X \times Y \mapsto \mathbb{R}$, we have the minimax inequality:
$$\sup_{x\in X}\inf_{y\in Y}\phi(x,y) \leq \inf_{y\in Y}\sup_{x\in X}\phi(x,y), \qquad (2)$$
Proof: To see this, for every $\bar{x} \in X$, write
$$\inf_{y\in Y}\phi(\bar{x},y) \leq \inf_{y\in Y}\sup_{x\in X}\phi(x,y)$$
and take the supremum over $\bar{x} \in X$ of the left-hand side.
- Eq. (1) implies that
$$\inf_{y\in Y}\sup_{x\in X} x^T Ay \leq \sup_{x\in X} x^T Ay^* = (x^*)^T Ay^* = \inf_{y\in Y}(x^*)^T Ay \leq \sup_{x\in X}\inf_{y\in Y} x^T Ay,$$
which combined with the minimax inequality [cf. Eq. (2)], proves that equality holds throughout in the preceding.
- Hence, a mixed strategy profile (x^*, y^*) is a Nash equilibrium if and only if
$$(x^*)^T Ay^* = \inf_{y\in Y}\sup_{x\in X} x^T Ay = \sup_{x\in X}\inf_{y\in Y} x^T Ay.$$
We refer to $(x^*)^T Ay^*$ as the value of the game.

Zero-Sum Finite Games

- We next show that finding the mixed strategy Nash equilibrium strategies and the value of the game can be written as a pair of linear optimization problems.
- For a fixed y, we have

$$\max_{x \in X} x^T Ay = \max_{i=1,\ldots,n} \{[Ay]_i\},$$

and therefore

$$\begin{aligned} \min_{y \in Y} \max_{x \in X} x^T AY &= \min_{y \in Y} \max\{[Ay]_1, \ldots, [Ay]_n\} \\ &= \min_{y \in Y,\ v\mathbf{1}_n \geq Ay} v. \end{aligned}$$

- Hence, the value of the game and the Nash equilibrium strategy of player 2 can be obtained as the optimal value and the optimal solution of the preceding linear optimization problem.

Zero-Sum Finite Games

- Similarly, we have

$$\begin{aligned}\max_{x\in X}\min_{y\in Y} x^T Ay &= \max_{x\in X}\min\{[A^T x]_1,\dots,[A^T x]_m\} \\ &= \max_{x\in X,\ \xi\mathbf{1}_m\le A^T x} \xi.\end{aligned}$$

- Linear optimization problems can be solved efficiently (in time polynomial in m and n).
- We next discuss alternative approaches for computing the mixed Nash equilibrium of two-player nonzero-sum finite games.

Nonzero-Sum Finite Games

Solution of Algebraic Equations:

- We first consider an inner or totally mixed Nash equilibrium (x^*, y^*), i.e., $x_i^* > 0$ for all i and $y_j^* > 0$ for all j (all pure strategies are used with positive probability).
- Let a_i denote the rows of payoff matrix A and b_j denote the columns of payoff matrix B.
- Using the equivalent characterization of a mixed strategy Nash equilibrium (i.e., all pure strategies in the support of a Nash equilibrium strategy yields the same payoff, which is also greater than or equal to the payoffs for strategies outside the support), we have

$$a_1 y^* = a_i y^*, \qquad i = 2, \ldots, n,$$
$$(x^*)^T b_1 = (x^*)^T b_j, \qquad j = 2, \ldots, m.$$

- The preceding is a system of linear equations which can be solved efficiently. (Note that for more than two players, we will have polynomial equations.)

Nonzero-Sum Finite Games

- However, the assumption that every strategy is played with positive probability is a very restrictive assumption. Most games do not have totally mixed Nash equilibria.
- For such games, we can use the preceding characterization to come up with a naive way to compute all the Nash equilibria of a finite two-player game: A mixed strategy profile $(x^*, y^*) \in X \times Y$ is a Nash equilibrium with support $\bar{S}_1 \subset S_1$ and $\bar{S}_2 \subset S_2$ if and only if

$$\begin{aligned} u = a_i y^*, \quad \forall\ i \in \bar{S}_1, \qquad & u \geq a_i y^*, \qquad \forall\ i \notin \bar{S}_1, \\ v = (x^*)^T b_j, \quad \forall\ j \in \bar{S}_2, \qquad & v \geq (x^*)^T b_j, \quad \forall\ j \notin \bar{S}_2, \\ x_i^* = 0, \quad \forall\ i \notin \bar{S}_1, \qquad & y_j^* = 0, \quad \forall\ j \notin \bar{S}_2. \end{aligned}$$

- To find the right supports for the above procedure to work, we need to search over all possible supports. Since there are 2^{n+m} different supports, this procedure leads to an exponential complexity in the number of pure strategies of the players.

Remark: Computational complexity of computing Nash equilibrium for finite games lies in finding the right support sets.

Nonzero-Sum Finite Games

Optimization Formulation:

- A general method for the solution of a bimatrix game is to transform it into a nonlinear (in fact, a bilinear) programming problem, and to use the techniques developed for solutions of nonlinear programming problems.

Proposition

A mixed strategy profile (x^*, y^*) is a mixed Nash equilibrium of the bimatrix game (A, B) if and only if there exists a pair (p^*, q^*) such that (x^*, y^*, p^*, q^*) is a solution to the following bilinear programming problem:

$$\begin{aligned} &\textit{maximize} && \{x^T Ay + x^T By - p - q\} && (3)\\ &\textit{subject to} && Ay \leq p\mathbf{1}_n, \quad B^T x \leq q\mathbf{1}_m, && (4)\\ & && \sum_i x_i = 1, \quad \sum_j y_j = 1, && \\ & && x \geq 0, \;\; y \geq 0, && (5) \end{aligned}$$

where $\mathbf{1}_n$ ($\mathbf{1}_m$) denotes the n (m)-dimensional vector with all components equal to 1.

Proof

- Assume first that (x^*, y^*) is a Nash equilibrium.
- For any feasible solution of problem (3) (x, y, p, q), the constraints (4) imply that
$$x^T Ay + x^T By - p - q \leq 0. \quad (6)$$
- Let $p^* = (x^*)^T Ay^*$ and $q^* = (x^*)^T By^*$. Then the vector (x^*, y^*, p^*, q^*) has an optimal value equal to 0. If the vector (x^*, y^*, p^*, q^*) is also feasible, it follows by Eq. (6) that it is an optimal solution of problem (3).
- Since (x^*, y^*) is a Nash equilibrium, we have
$$(x^*)^T Ay^* \geq x^T Ay^*, \qquad \forall x \in X.$$
- Choosing $x = e_i$, i.e., the i^{th} unit vector, which has all 0s except a 1 in the i^{th} component, we obtain
$$p^* = (x^*)^T Ay^* \geq [Ay^*]_i,$$
for each i, showing that (x^*, y^*, p^*, q^*) satisfies the first constraint in (4).
- The fact that it satisfies the second constraint can be shown similarly, hence proving that (x^*, y^*, p^*, q^*) is an optimal solution of problem (3).

Proof

- Conversely, assume that $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ is an optimal solution of problem (3).
- Since all feasible solutions have nonpositive optimal value [see Eq. (6)], and any mixed strategy Nash equilibrium (which exists by Nash's Theorem) was shown to have an optimal value equal to 0, it follows that

$$\bar{x}^T A\bar{y} + \bar{x}^T B\bar{y} - \bar{p} - \bar{q} = 0. \tag{8}$$

- For any $x \geq 0$ with $\sum_i x_i = 1$ and $y \geq 0$ with $\sum_j y_j = 1$, the constraints in (4) imply that

$$x^T A\bar{y} \leq \bar{p},$$
$$y^T B^T \bar{x} \leq \bar{q}.$$

- In particular, we have $\bar{x}^T A\bar{y} \leq \bar{p}$ and $\bar{y}^T B^T \bar{x} \leq \bar{q}$. Combining with Eq. (8), we obtain $\bar{x}^T A\bar{y} = \bar{p}$ and $\bar{y}^T B^T \bar{x} = \bar{q}$.
- Together with the preceding set of equations, this yields

$$x^T A\bar{y} \leq \bar{x}^T A\bar{y}, \qquad \text{for all } x \in X,$$
$$y^T B^T \bar{x} \leq \bar{y}^T B^T \bar{x}, \qquad \text{for all } y \in Y.$$

Nonzero-Sum Finite Games

Linear Complementarity Problem Formulation

- Recall that a_i denotes the rows of the payoff matrix of player 1 A, and b_j denotes the columns of the payoff matrix of player 2.
- Then, the mixed strategy profile $(x, y) \in X \times Y$ is a Nash equilibrium if and only if

$$x_i > 0 \quad \rightarrow \quad a_i y = \max_k a_k y,$$
$$y_j > 0 \quad \rightarrow \quad x^T b_j = \max_k x^T b_k.$$

- By introducing the additional variables $r_i \in \mathbb{R}^n$, $r_i \geq 0$ for $i = 1, 2$ (i.e., *slack* variables), and $v_i \in \mathbb{R}$, for $i = 1, 2$, we can write the preceding equivalently as

$$Ay + r_1 = v_1 \mathbf{1}_n,$$
$$B^T x + r_2 = v_2 \mathbf{1}_m,$$
$$x^T r_1 = 0, \qquad y^T r_2 = 0.$$

  Since $x \geq 0$, $y \geq 0$, and $r_i \geq 0$, the last equation also implies that $x_1 r_{1i} = 0$ for all $i = 1, \ldots, n$ and $y_j r_{2,j} = 0$.

Nonzero-Sum Finite Games

- Assume now that $v_1 > 0$ and $v_2 > 0$ (which holds if all components of A and B are positive).
- We normalize the variables y and r_1 by v_1, and x and r_2 by v_2 and use the notation

$$z = [x, y]^T, \quad r = [r_1, r_2]^T, \quad q = [\mathbf{1}_n, \mathbf{1}_m]^T,$$
$$U = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}.$$

- If we further drop the constraints $\sum_i x_i = 1$ and $\sum_j y_j = 1$ (at the expense of having an additional extraneous solution $z = [0, 0]^T$), we obtain the following linear complementarity problem formulation

$$Uz + r = q, \qquad z \geq 0, \; r \geq 0, \tag{9}$$
$$z^T r = 0.$$

- The last condition is referred to as the complementary slackness or the complementarity condition.

Extensions

- The formulations for nonzero-sum games we have discussed before can be generalized to multiple-player finite games.
- Recent work [Parrilo 06] has focused on two person zero-sum games with continuous strategy spaces and some structure on the payoff functions, and has shown that the equilibrium strategies and the value of the game can be obtained efficiently.
- Some of these results were extended by [Stein, Ozdaglar, Parrilo 08] to nonzero sum games.

Computing Approximate Nash Equilibria

- We next study a quasi-polynomial algorithm for computing an ϵ-Nash equilibrium.
- We follow the development of [Lipton, Markakis, and Mehta 03].
- Our focus will be on games with two players, in which both players have the same number of strategies n. We denote the $n \times n$ payoff matrices of players 1 and 2 by A and B, respectively.
- The next definition captures the notion of "simple mixed strategies".

Definition

A mixed strategy of player i is called **k-uniform** *if it is the uniform distribution on a subset $\bar{S}_i$ of the pure strategies S_i with $|\bar{S}_i| = k$.*

For example, for a player with 3 pure strategies both $x = [1/3, 1/3, 1/3]$ and $x = [2/3, 1/3, 0]$ are 3-uniform strategies.

Computing Approximate Nash Equilibria

Recall the definition of an ϵ-equilibrium.

Definition

Given some scalar $\epsilon > 0$, a mixed strategy profile $(\bar{x}, \bar{y})$ is an ϵ-equilibrium if

$$x^T A\bar{y} \leq \bar{x}^T A\bar{y} + \epsilon \qquad \text{for all } x \in X,$$

$$\bar{x}^T By \leq \bar{x}^T B\bar{y} + \epsilon \qquad \text{for all } y \in Y.$$

The next theorem presents the main result.

Computing Approximate Nash Equilibria

Theorem

Assume that all the entries of the matrices A and B are between 0 and 1. Let (x^*, y^*) be a mixed Nash equilibrium and let $\epsilon > 0$. For all $k \geq \frac{32 \log n}{\epsilon^2}$, there exists a pair of k-uniform strategies $(\bar{x}, \bar{y})$ such that

- *$(\bar{x}, \bar{y})$ is an ϵ-equilibrium.*
- *$\left|\bar{x}^T A\bar{y} - (x^*)^T Ay^*\right| < \epsilon$, i.e., player 1 gets almost the same payoff as in the Nash equilibrium.*
- *$\left|\bar{x}^T B\bar{y} - (x^*)^T By^*\right| < \epsilon$, i.e., player 2 gets almost the same payoff as in the Nash equilibrium.*

The proof relies on a probabilistic sampling argument. This theorem establishes the existence of a $k - uniform$ mixed strategy profile $(\bar{x}, \bar{y})$, which not only forms an ϵ-Nash equilibrium, but also provide both players a payoff ϵ close to the payoffs they would obtain at some Nash equilibrium.

Computing Approximate Nash Equilibria

Corollary

Consider a 2-player game with n pure strategies for each player. There exists an algorithm that is quasi-polynomial in n for computing an ϵ-Nash equilibrium.

- Let $k \geq \frac{32 logn}{\epsilon^2}$.
- By an exhaustive search, we can find all $k - uniform$ mixed strategies for each player.
- Verifying ϵ-equilibrium condition is easy since we need to check only deviations to pure strategies.
- The running time of the algorithm is quasi-polynomial, i.e., $n^{O(\log n)}$ since there are $\binom{n+k-1}{k}^2 \approx n^k$ possible pairs of k-uniform strategies.

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