6.453 Quantum Optical Communication
Spring 2009

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Lecture Number 4
Fall 2008

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Date: Tuesday, September 16, 2008

Reading: For the quantum harmonic oscillator and its energy eigenkets:

- C.C. Gerry and P.L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005) pp. 10–15.
- W.H. Louisell, *Quantum Statistical Properties of Radiation* (McGraw-Hill, New York, 1973) sections 2.1–2.5.
- R. Loudon, *The Quantum Theory of Light* (Oxford University Press, Oxford, 1973) pp. 128–133.

Introduction

In Lecture 3 we completed the foundations of Dirac-notation quantum mechanics. Today we'll begin our study of the quantum harmonic oscillator, which is the quantum system that will pervade the rest of our semester's work. We'll start with a classical physics treatment and—because 6.453 is an Electrical Engineering and Computer Science subject—we'll develop our results from an LC circuit example.

Classical LC Circuit

Consider the undriven LC circuit shown in Fig. 1. As in Lecture 2, we shall take the state variables for this system to be the charge on its capacitor, $q(t) = Cv(t)$, and the flux through its inductor, $p(t) \equiv Li(t)$. Furthermore, we'll consider the behavior of this system for $t \geq 0$ when one or both of the initial state variables are non-zero, i.e., $q(t) \neq 0$ and/or $p(0) \neq 0$. You should already know that this circuit will then undergo simple harmonic motion, i.e., the energy stored in the circuit will slosh back and forth between being electrical (stored in the capacitor) and magnetic (stored in the inductor) as the voltage and current oscillate sinusoidally at the resonant frequency $\omega = 1/\sqrt{LC}$. Nevertheless, we shall develop that behavior here to make it explicit for our use in establishing the quantum theory of harmonic oscillation.

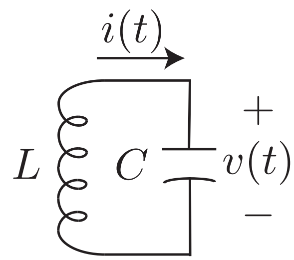

Figure 1: The undriven LC circuit: a classical harmonic oscillator.

The equations of motion for our LC circuit are

$$\dot{q}(t) \equiv \frac{\mathrm{d}q(t)}{\mathrm{d}t} = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} = i(t) = \frac{p(t)}{L} \tag{1}$$

$$\dot{p}(t) \equiv \frac{\mathrm{d}p(t)}{\mathrm{d}t} = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = -v(t) = -\frac{q(t)}{C}. \tag{2}$$

The energy stored in the capacitor and the inductor are, respectively, $Cv^2(t)/2$ and $Li^2(t)/2$, so that the total energy (the Hamiltonian) for the circuit is

$$H = \frac{Cv^2(t)}{2} + \frac{Li^2(t)}{2} = \frac{q^2(t)}{2C} + \frac{p^2(t)}{2L}. \tag{3}$$

Note that H is a constant of the motion, because the LC circuit is both undriven and undamped, i.e., it is passive and lossless.

At this point we can reap the benefit of our particular choice for the state variables in that Eqs. (1) and (2) can be written in the canonical form of Hamilton's equations. In particular, writing $H = H(q,p)$, we have that

$$\frac{\partial H(q,p)}{\partial q} = -\dot{p}(t) \quad \text{and} \quad \frac{\partial H(q,p)}{\partial p} = \dot{q}(t), \tag{4}$$

for our LC circuit, as you can (and should) verify.

Now, let us examine the time evolution of the state variables. Differentiating (1) and employing (2) we find that

$$\ddot{q}(t) = \frac{\dot{p}(t)}{L} = -\frac{q(t)}{LC}, \tag{5}$$

from which it follows that

$$q(t) = \mathrm{Re}(\mathbf{q}e^{-j\omega t}), \quad \text{for } t \geq 0, \tag{6}$$

where

$$\omega = 1/\sqrt{LC} \tag{7}$$

is the circuit's resonant frequency and $\mathbf{q}$ must be determined from the given initial conditions $q(0)$ and $p(0)$. The first of these initial conditions immediately tells us that

$$\mathrm{Re}(\mathbf{q}) = q(0). \tag{8}$$

To satisfy the second initial condition, we employ (1) to obtain

$$p(t) = L\dot{q}(t) = \mathrm{Re}(-j\omega L\mathbf{q}e^{-j\omega t}) = \omega L \mathrm{Im}(\mathbf{q}e^{-j\omega t}), \quad \text{for } t \geq 0, \tag{9}$$

so that

$$\mathrm{Im}(\mathbf{q}) = \frac{p(0)}{\omega L}. \tag{10}$$

Evaluating the total energy, in terms of these solutions for $q(t)$ and $p(t)$, then gives

$$H = \frac{q^2(t)}{2C} + \frac{p^2(t)}{2L} = \frac{[\mathrm{Re}(\mathbf{q}e^{-j\omega t})]^2}{2C} + \frac{[\omega L \mathrm{Im}(\mathbf{q}e^{-j\omega t})]^2}{2L} = \frac{|\mathbf{q}|^2}{2C}, \tag{11}$$

which is a constant, as expected.

The state variables appearing in Hamilton's equations are called canonically conjugate variables. In our LC example they have physical units, e.g., $q(t)$ is measured in Coulombs. For our future purposes, it is much more convenient to work with dimensionless quantities. To do so for our LC circuit, and without loss of generality, we shall assume that $L = 1$ and define new normalized variables

$$a_1(t) \equiv \sqrt{\frac{\omega}{2\hbar}}\, q(t), \quad a_2(t) \equiv \sqrt{\frac{1}{2\hbar\omega}}\, p(t), \quad a(t) \equiv a_1(t) + ja_2(t), \tag{12}$$

where $\hbar$ is Planck's constant divided by 2π. Our solutions for $q(t)$ and $p(t)$ then become the following results for $a_1(t)$ and $a_2(t)$,

$$a_1(t) = \mathrm{Re}\left[\sqrt{\frac{\omega}{2\hbar}}\,\mathbf{q}e^{-j\omega t}\right] \quad \text{and} \quad a_2(t) = \mathrm{Im}\left[\sqrt{\frac{1}{2\hbar\omega}}\,\omega\mathbf{q}e^{-j\omega t}\right], \quad \text{for } t \geq 0. \tag{13}$$

From these results we can write

$$a(t) = ae^{-j\omega t}, \quad \text{where } a = a(0) = \sqrt{\frac{\omega}{2\hbar}}\,\mathbf{q}, \tag{14}$$

and

$$H = \frac{2\hbar}{\omega}\frac{a_1^2(t)}{2C} + 2\hbar\omega\frac{a_2^2(t)}{2} = \hbar\omega[a_1^2(t) + a_2^2(t)] = \hbar\omega|a(t)|^2 = \hbar\omega|a|^2, \tag{15}$$

where we have used $L = 1$ and $\omega = 1/\sqrt{C}$.

Quantum LC Circuit

With the dimensionless reformulation of the classical LC circuit in hand, we are ready to begin the quantum treatment of this system. Dirac taught us that if we have a classical physical system—such as the LC circuit that we considered in the previous section—governed by Hamilton's equations, then we quantize this system by converting the Hamiltonian, H, and the canonical variables, $q(t)$ and $p(t)$, into (Heisenberg picture) observables $\hat{H}$, $\hat{q}(t)$, and $\hat{p}(t)$, respectively, with the latter two having the non-zero commutator[1]

$$[\hat{q}(t), \hat{p}(t)] = j\hbar. \tag{16}$$

From Lecture 3, we know that this non-zero commutator implies that we cannot simultaneously measure the capacitor charge and inductor flux in the quantized LC circuit, i.e., we have that

$$\langle \Delta\hat{q}^2(t)\rangle\langle \Delta\hat{p}^2(t)\rangle \geq \hbar^2/4, \tag{17}$$

from the Heisenberg uncertainty principle.

In our dimensionless reformulation we have that the Hamiltonian satisfies

$$\hat{H} = \hbar\omega[\hat{a}_1^2(t) + \hat{a}_2^2(t)], \tag{18}$$

where $\hat{a}_1(t)$ and $\hat{a}_2(t)$ are the dimensionless observables that take the place—in the quantum treatment—of $a_1(t)$ and $a_2(t)$ from the classical case. We also define $\hat{a}(t) = \hat{a}_1(t) + j\hat{a}_2(t)$, in analogy with the classical case, but, as we now show, due care must be taken in writing $\hat{H}$ in terms of $\hat{a}(t)$, because $\hat{a}_1(t)$ and $\hat{a}_2(t)$ do *not* commute.

From (16), and the definitions in our dimensionless reformulation, we have that[2]

$$[\hat{a}_1(t), \hat{a}_2(t)] = \left[\sqrt{\frac{\omega}{2\hbar}}\,\hat{q}(t), \sqrt{\frac{1}{2\hbar\omega}}\,\hat{p}(t)\right] = \frac{[\hat{q}(t), \hat{p}(t)]}{2\hbar} = \frac{j}{2}, \tag{19}$$

so that these dimensionless operators are non-commuting observables. The Heisenberg uncertainty principle for these observables then takes the form

$$\langle \Delta\hat{a}_1^2(t)\rangle\langle \Delta\hat{a}_2^2(t)\rangle \geq 1/16, \tag{20}$$

and will be the focus of much of our work this semester.

Note that $\hat{a}(t)$ is *not* an Hermitian operator, because

$$\hat{a}^\dagger(t) = \hat{a}_1^\dagger(t) - j\hat{a}_2^\dagger(t) = \hat{a}_1(t) - j\hat{a}_2(t) \neq \hat{a}(t). \tag{21}$$

[1]The right-hand side of this equation is really an operator, i.e., it is $j\hbar\hat{I}$, where $\hat{I}$ is the identity operator. It is customary, however, to suppress the $\hat{I}$ in this expression.

[2]Once again, a factor of $\hat{I}$ has been omitted from the right-hand side. Going forward, we will not make further note of such omissions. Any purely classical term in an operator-valued equation should be interpreted as having an implicit factor of $\hat{I}$.

For the classical LC circuit, $a(t)$ is a complex-valued function that completely characterizes the time-evolution of the system, and, moreover, its squared magnitude—which is a constant of the motion—is proportional to the energy in the circuit. For our quantum LC circuit, however, things are more complicated with respect to $\hat{a}(t)$ and its role in the Hamiltonian (energy) operator $\hat{H}$. Because $\hat{a}_1(t)$ and $\hat{a}_2(t)$ do not commute, we find that this is also the case for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$,

$$[\hat{a}(t), \hat{a}^\dagger(t)] = [\hat{a}_1(t)+j\hat{a}_2(t), \hat{a}_1(t)-j\hat{a}_2(t)] = -j[\hat{a}_1(t), \hat{a}_2(t)]+j[\hat{a}_2(t), \hat{a}_1(t)] = 1. \quad (22)$$

Using this commutator plus

$$\hat{a}_1(t) = \frac{\hat{a}(t) + \hat{a}^\dagger(t)}{2} \quad \text{and} \quad \hat{a}_2(t) = \frac{\hat{a}(t) - \hat{a}^\dagger(t)}{2j} \quad (23)$$

in our previous expression for the Hamiltonian, $\hat{H}$, we get

$$\hat{H} = \hbar\omega(\hat{a}^\dagger(t)\hat{a}(t) + 1/2). \quad (24)$$

The Heisenberg equation of motion for $\hat{a}(t)$ is therefore as follows,

$$j\hbar\frac{\mathrm{d}\hat{a}(t)}{\mathrm{d}t} = [\hat{a}(t), \hat{H}] = \hbar\omega[\hat{a}(t), \hat{a}^\dagger(t)\hat{a}(t) + 1/2] \quad (25)$$

$$= \hbar\omega[\hat{a}(t)\hat{a}^\dagger(t)\hat{a}(t) - \hat{a}^\dagger(t)\hat{a}^2(t)] = \hbar\omega\hat{a}(t), \quad \text{for } t \geq 0. \quad (26)$$

Not surprisingly, the solution to this equation is exactly what we expect for simple harmonic motion:

$$\hat{a}(t) = \hat{a}e^{-j\omega t}, \quad \hat{a}_1(t) = \mathrm{Re}(\hat{a}e^{-j\omega t}), \quad \hat{a}_2(t) = \mathrm{Im}(\hat{a}e^{-j\omega t}), \quad \text{for } t \geq 0, \quad (27)$$

where $\hat{a} = \hat{a}(0)$ is the initial condition. We also get

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2), \quad (28)$$

affirming that the energy—in our quantum LC circuit—is indeed a constant of the motion.

The term $\hbar\omega/2$ that appears in (28) is of great physical significance. Let $|\psi\rangle$ be an arbitrary state of the quantum harmonic oscillator, we have that its average energy satisfies

$$\langle\psi|\hat{H}|\psi\rangle = \hbar\omega(\langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle + 1/2) \geq \hbar\omega/2, \quad (29)$$

so that there is *always* non-zero average energy in the oscillator, whereas in the classical case we would have $H = 0$ when there is no charge on the capacitor and no flux through the inductor. The term $\hbar\omega/2$ is thus called the *zero-point energy* of the oscillator, about which we will learn more later. For now, however, let's use the Heisenberg uncertainty principle to contrast the simple harmonic motion in the quantum world with what occurs in classical physics.

The left-hand side of Slide 8 shows a phasor picture and time-domain plot for a classical harmonic oscillator. Here, there is no fundamental requirement of randomness, i.e., the initial condition on $a(t)$ can be a point in phase space, and a plot of $a_1(t)$ versus t is a noiseless sinusoid. What happens in the quantum world is different. Suppose that the oscillator is in a quantum state $|\psi\rangle$ such that $\langle\psi|\hat{a}|\psi\rangle = a$, where a is the initial condition of the classical oscillator, and

$$\langle\Delta\hat{a}_1^2(t)\rangle = \langle\Delta\hat{a}_2^2(t)\rangle = 1/4, \tag{30}$$

so that $|\psi\rangle$ is a minimum uncertainty-product state for the Heisenberg inequality (20) with *equal* uncertainties in $\hat{a}_1(t)$ and $\hat{a}_2(t)$.[3] The right-hand side of Slide 8 is a qualitative representation of this state's behavior. The phasor picture is a circular contour enclosing one standard deviation from the mean of $\langle\psi|\mathrm{Re}(\hat{a}e^{j\theta})|\psi\rangle$ for $0 \leq \theta \leq 2\pi$. The three time-domain curves are the mean and the mean $\pm$ one standard deviation plots for the $\hat{a}_1(t)$ measurement when the system is in state $|\psi\rangle$.[4] In summary, the classical oscillator can be noiseless, but the quantum oscillator is fundamentally noisy.

The Energy Eigenkets

To progress further we need to develop some specific kets that characterize the quantum harmonic oscillator. We'll begin, in today's lecture, with the energy eigenkets. We have already seen that the oscillator has a minimum average energy that is at least $\hbar\omega/2$. If there is a minimum-energy state of the oscillator, $|E_0\rangle$, such that $\langle E_0|\hat{H}|E_0\rangle = \hbar\omega/2$, then it must satisfy

$$\hat{a}|E_0\rangle = 0 \quad \text{and} \quad \hat{H}|E_0\rangle = \frac{\hbar\omega}{2}|E_0\rangle, \tag{31}$$

i.e., it is the minimum energy eigenket of $\hat{H}$ with eigenvalue $E_0 = \hbar\omega/2$. Let us *assume* that such an eigenket exists, and try to find other eigenkets $\{|E_n\rangle\}$ and eigenvalues $\{E_n\}$ such that

$$\hat{H}|E_n\rangle = E_n|E_n\rangle, \quad \text{for } n = 1, 2, \ldots, \tag{32}$$

where $E_0 < E_1 < E_2 < \cdots$.[5]

Consider the behavior of the ket $\hat{a}|E_n\rangle$. We have that

$$\hat{H}(\hat{a}|E_n\rangle) = \hbar\omega(\hat{a}^\dagger\hat{a}^2 + \hat{a}/2)|E_n\rangle = \hbar\omega(\hat{a}\hat{a}^\dagger\hat{a} - \hat{a} + \hat{a}/2)|E_n\rangle \tag{33}$$

$$= \hat{a}\hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)|E_n\rangle - \hbar\omega\hat{a}|E_n\rangle = (E_n - \hbar\omega)(\hat{a}|E_n\rangle), \tag{34}$$

[3] We shall construct such a state in Lecture 5. For now, just assume that such a state exists.

[4] This means that the values of these curves at any particular time $t = t_0$ are the mean and the mean $\pm$ one standard deviation for the measurement $\hat{a}_1(t_0)$. Because of the projection postulate, these curves do *not* represent the behavior of a continuous-time measurement of $\hat{a}_1(t)$.

[5] Here we have taken the liberty of assuming that the Hamiltonian for the oscillator has a discrete eigenspectrum.

so that $\hat{a}|E_n\rangle$ is an eigenket of $\hat{H}$ with eigenvalue $E_n - \hbar\omega$. Likewise, if we consider the ket $\hat{a}^\dagger|E_n\rangle$, we find that

$$\hat{H}(\hat{a}^\dagger|E_n\rangle) = \hbar\omega(\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}^\dagger/2)|E_n\rangle = \hbar\omega(\hat{a}^{\dagger 2}\hat{a} + \hat{a}^\dagger + \hat{a}^\dagger/2)|E_n\rangle \tag{35}$$

$$= \hat{a}^\dagger\hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)|E_n\rangle + \hbar\omega\hat{a}^\dagger|E_n\rangle = (E_n + \hbar\omega)(\hat{a}^\dagger|E_n\rangle), \tag{36}$$

which shows that $\hat{a}^\dagger|E_n\rangle$ is an eigenket of $\hat{H}$ with eigenvalue $E_n + \hbar\omega$.

We can now easily complete our derivation of the energy eigenkets. The Hamiltonian is an observable, so it must have a complete set of orthonormal eigenkets. We have shown that applying $\hat{a}$ to the eigenket $|E_n\rangle$ with eigenvalue E_n results in another eigenket with eigenvalue $E_n - \hbar\omega$. Applying $\hat{a}$ to $\hat{a}|E_n\rangle$ then yields yet another eigenket, this time with eigenvalue $E_n - 2\hbar\omega$. The next application of $\hat{a}$ produces an eigenket with eigenvalue $E_n - 3\hbar\omega$, etc. But this sequence must terminate, because the oscillator has a minimum energy of $\hbar\omega/2$. It follows that

$$E_n = \hbar\omega(n + 1/2), \quad \text{for } n = 0, 1, 2, \ldots, \tag{37}$$

must be the eigenvalues of $\hat{H}$. In words, this says that the energy of the oscillator is quantized, with its possible values being n times $\hbar\omega$ plus the zero-point energy. Anticipating that the quantum harmonic oscillator represents a single-mode electromagnetic field, let's refer to these energy quanta as *photons*.

It is convenient, at this juncture, to introduce a (photon) number operator,

$$\hat{N} \equiv \hat{a}^\dagger\hat{a}, \tag{38}$$

so that $\hat{H} = \hbar\omega(\hat{N}+1/2)$. The number operator is Hermitian and it should be easy for you to verify that its eigenkets $\{|n\rangle\}$ are the energy eigenkets, viz. $|n\rangle = |E_n\rangle$, and its associated eigenvalues are the non-negative integers. From our work on observables, we then have that

$$\hat{N} = \sum_{n=0}^{\infty} n|n\rangle\langle n| \quad \text{and} \quad \hat{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{39}$$

with $\langle n|m\rangle = \delta_{nm}$. It will be important for us to represent $\hat{a}$ and $\hat{a}^\dagger$ in terms of the number kets $\{|n\rangle\}$. We know that

$$\hat{a}|n\rangle = c_n|n-1\rangle, \tag{40}$$

but we need to find out the value of the constant c_n. Because

$$|c_n|^2 = (\hat{a}|n\rangle)^\dagger(\hat{a}|n\rangle) = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle n|\hat{N}|n\rangle = n, \tag{41}$$

we shall take $c_n = \sqrt{n}$, by assuming that this constant is positive real, and get

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \quad \text{for } n = 1, 2, 3, \ldots \tag{42}$$

Similarly, we know that

$$\hat{a}^\dagger|n\rangle = d_n|n+1\rangle, \tag{43}$$

and we determine d_n via

$$|d_n|^2 = (\hat{a}^\dagger|n\rangle)^\dagger(\hat{a}^\dagger|n\rangle_{=}\langle n|\hat{a}\hat{a}^\dagger|n\rangle = n+1, \tag{44}$$

whence $d_n = \sqrt{n+1}$ under the assumption that this constant is positive real. Thus, we have

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle, \quad \text{for } n = 0, 1, 2, \ldots \tag{45}$$

Equations (42) and (45), respectively, show that the operators $\hat{a}$ and $\hat{a}^\dagger$ annihilate and create energy quanta (photons) of the quantum harmonic oscillator. Hence they are called the (photon) annihilation and (photon) creation operators. The real and imaginary parts of $\hat{a}$ are called its quadrature components, in analogy with the terminology for the phasor a of the classical harmonic oscillator. Equations (42) and (45)—plus the orthonormality of the number kets $\{|n\rangle\}$—can be used to show that

$$\hat{a} = \sum_{n=1}^{\infty}\sqrt{n}\,|n-1\rangle\langle n| \quad \text{and} \quad \hat{a}^\dagger = \sum_{n=0}^{\infty}\sqrt{n+1}\,|n+1\rangle\langle n|, \tag{46}$$

but the proof is left as an exercise for the reader.

The Road Ahead

In the next lecture we shall continue our development of the quantum harmonic oscillator. We shall study the quadrature-measurement statistics of the number kets, and introduce the coherent states of the oscillator. The latter are minimum uncertainty-product states for the quadratures with equal uncertainties in each quadrature. They are also the states that give rise to the classical noiseless oscillation in the limit of infinite excitation.