6.453 Quantum Optical Communication
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Reading: For electromagnetic field quantization:

- W.H. Louisell, *Quantum Statistical Properties of Radiation* (McGraw-Hill, New York, 1973) sections 4.3, 4.4.
- L. Mandel and E. Wolf *Optical Coherence and Quantum Optics,* (Cambridge University Press, Cambridge, 1995) sections 10.1–10.3.

Introduction

Today we move on to the final section of material on quantum optical communication: the full multi-temporal mode treatments of electromagnetic field quantization, photodetection theory, nonlinear optics, non-classical light generation, and quantum interference. Because our prerequisite subjects—6.011 and 18.06—do not include enough background for these topics, we'll tread gently. Hence there will not be any more problem sets. (On the other hand, you will need the freed-up time to do the reading for and preparation of your term papers.)

Classical Electromagnetic Waves in Free Space

Before we can quantize the electromagnetic field, we must develop some understanding of the classical electromagnetic field.

From Maxwell's Equation to the Wave Equation

Consider a region of empty space in which there is no charge density and no current density, i.e., it is source free. Classical electromagnetism within such a region is governed by the source-free version of Maxwell's equations with the vacuum constitutive relations. In differential form these equations are as follows:

$$\nabla \times \vec{E}(\vec{r},t) = -\mu_0 \frac{\partial}{\partial t}\vec{H}(\vec{r},t) \quad \text{and} \quad \nabla \cdot \epsilon_0 \vec{E}(\vec{r},t) = 0 \tag{1}$$

$$\nabla \times \vec{H}(\vec{r},t) = \epsilon_0 \frac{\partial}{\partial t}\vec{E}(\vec{r},t) \quad \text{and} \quad \nabla \cdot \mu_0 \vec{H}(\vec{r},t) = 0, \tag{2}$$

where $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ are the electric and magnetic fields (units V/m and A/m), and ϵ_0 and μ_0 are the permittivity and permeability of free space. The curl and divergence equations for the electric field are Faraday's law and Gauss' law, and the curl equation for the magnetic field is Ampère's law.

For our purposes, it is convenient to work in the Coulomb gauge, i.e., we introduce a vector potential $\vec{A}(\vec{r},t)$ that is divergence free, $\nabla \cdot \vec{A}(\vec{r},t) = 0$. Then, if we take

$$\vec{E}(\vec{r},t) \equiv -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} \quad \text{and} \quad \vec{H}(\vec{r},t) \equiv \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r},t), \tag{3}$$

we find that

$$\nabla \cdot \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t} \nabla \cdot \vec{A}(\vec{r},t) = 0, \tag{4}$$

because of the Coulomb gauge condition, and

$$\nabla \cdot \vec{H}(\vec{r},t) = \frac{1}{\mu_0} \nabla \cdot [\nabla \times \vec{A}(\vec{r},t)] = 0, \tag{5}$$

because of the vector calculus identity $\nabla \cdot [\nabla \times \vec{F}(\vec{r},t)] = 0$, for any $\vec{F}(\vec{r},t)$. In addition, (3) gives us

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t} \nabla \times \vec{A}(\vec{r},t) = -\mu_0 \frac{\partial \vec{H}(\vec{r},t)}{\partial t}. \tag{6}$$

Thus, in Coulomb gauge, with the electric and magnetic fields derived from the vector potential via (3), we automatically satisfy three of Maxwell's four equations for a source-free region of free space. All we need do now is to determine how to satisfy Ampère's law.

To see what equation the vector potential must satisfy to ensure that the electric and magnetic fields obey Ampère's law, we substitute (3) into the left-hand side of Ampère's law, obtaining

$$\nabla \times \vec{H}(\vec{r},t) = \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A}(\vec{r},t) = \frac{1}{\mu_0} \{\nabla[\nabla \cdot \vec{A}(\vec{r},t)] - \nabla^2 \vec{A}(\vec{r},t)\} \tag{7}$$

$$= -\frac{1}{\mu_0} \nabla^2 \vec{A}(\vec{r},t), \tag{8}$$

where the second equality is a vector-calculus identity, and the third equality follows from the Coulomb gauge condition. Then, substituting (3) into the right-hand side of Ampère's law, we see that Ampère's law will be satisfied if

$$\nabla^2 \vec{A}(\vec{r},t) = -\mu_0 \epsilon_0 \frac{\partial \vec{E}(\vec{r},t)}{\partial t} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2}. \tag{9}$$

Rearranging this equation leads to the 3-D vector wave equation,

$$\nabla^2 \vec{A}(\vec{r},t) - \frac{1}{c^2} \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = \vec{0}, \tag{10}$$

where $c \equiv 1/\sqrt{\mu_0 \epsilon_0}$ has the units m/s, i.e., it is the speed of light.

Classical Plane-Wave Fields

We have just seen that the electric and magnetic fields in a source-free region of free space can be specified in terms of a Coulomb-gauge vector potential according to (3), and Maxwell's equations will be satisfied if the vector potential satisfies (10).[1] This equation is equivalent to three scalar differential equations, one for each Cartesian component of $\vec{A}(\vec{r},t)$, i.e.,

$$\nabla^2 A_k(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 A_k(\vec{r},t)}{\partial t^2} = 0, \quad \text{for } k = x,y,z. \tag{11}$$

That each one of these Cartesian-component equations is a wave equation can be seen by the following simple special case. Suppose that the x component of the vector potential is only a function of z and t. You should then be able to verify that

$$A_x(z,t) = f(t - z/c) + g(t + z/c) \tag{12}$$

is a solution to (11), with $f(\cdot)$ and $g(\cdot)$ being arbitrary real-valued functions. Moreover, the term $f(t-z/c)$ represents a $+z$-going wave moving at speed c, because at every constant-z plane the *same* pulse shape $f(t)$ appears delayed by z/c. Likewise, $g(t+z/c)$ represents a $-z$-going wave moving at speed c.

In order to quantize the electromagnetic field in a source-free region of free space, we need to have a general form for the solution to (10). To get such a general form we shall employ separation of variables, i.e., we shall seek solutions to (10) in which

$$\vec{A}(\vec{r},t) = \frac{1}{2\sqrt{\epsilon_0}}\sum_{\vec{l},\sigma} q_{\vec{l},\sigma}(t)\vec{u}_{\vec{l},\sigma}(\vec{r}) + \text{cc}, \tag{13}$$

where cc denotes complex conjugate Here, the vector potential has been written in terms of a collection of complex-valued modes, $\{q_{\vec{l},\sigma}(t)\vec{u}_{\vec{l},\sigma}(\vec{r})\}$, in which the time and space dependencies factor apart. These modes are indexed by the three-dimensional vector, $\vec{l}$, which will see later specifies the direction of propagation, and a scalar, σ, which we will see later specifies the polarization state of the mode. Because the 3D wave equation for $\vec{A}(\vec{r},t)$ is linear, each complex-valued term on the right in (13) must satisfy a 3D wave equation, i.e.,

$$\nabla^2[q_{\vec{l},\sigma}(t)\vec{u}_{\vec{l},\sigma}(\vec{r})] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[q_{\vec{l},\sigma}(t)\vec{u}_{\vec{l},\sigma}(\vec{r})] = \vec{0}. \tag{14}$$

This equation reduces to

$$[\nabla^2 \vec{u}_{\vec{l},\sigma}(\vec{r})]q_{\vec{l},\sigma}(t) - \frac{1}{c^2}\vec{u}_{\vec{l},\sigma}(\vec{r})\frac{\mathrm{d}^2 q_{\vec{l},\sigma}(t)}{\mathrm{d}t^2} = \vec{0}. \tag{15}$$

[1]One solution to this equation is the trivial one, $\vec{A}(\vec{r},t) = \vec{0}$. Our interest, however, is in non-trivial (non-zero) solutions. You might well ask how can their be a non-zero $\vec{A}(\vec{r},t)$ if there are no sources. The answer is that the sources which create this vector potential lie outside the region in which we are examining the electromagnetic field.

For there to be the assumed separation of variables, we must have that

$$\frac{\mathrm{d}^2 q_{\vec{l},\sigma}(t)}{\mathrm{d}t^2} \propto q_{\vec{l},\sigma}(t), \tag{16}$$

where the proportionality constant is *independent* of space and time. Anticipating future results—and being mindful of the units—we shall assume that this proportionality constant is a negative quantity $-\omega_{\vec{l}}^2$, so that[2]

$$\frac{\mathrm{d}^2 q_{\vec{l},\sigma}(t)}{\mathrm{d}t^2} + \omega_{\vec{l}}^2 q_{\vec{l},\sigma}(t) = 0, \tag{17}$$

and hence, by substitution of (17) into (15),

$$\nabla^2 \vec{u}_{\vec{l},\sigma}(\vec{r}) + \frac{\omega_{\vec{l}}^2}{c^2}\,\vec{u}_{\vec{l},\sigma}(\vec{r}) = \vec{0}. \tag{18}$$

At this point, we should begin to feel comfortable with statements made earlier in the term about each mode of an electromagnetic field being a harmonic oscillator: Eq. (17) shows that $q_{\vec{l},\sigma}(t)$ obeys the differential equation for frequency-$\omega_{\vec{l}}$ simple harmonic motion.[3] Equation (18) is called the Helmholtz equation; it governs the spatial characteristics of the mode indexed by $\vec{l}$ and σ. The key question that remains—insofar as this classical separation of variables is concerned—is how to determine the separation constant $\omega_{\vec{l}}$ for each mode. The answer is that this constant depends on the boundary conditions for the source-free region of free space that is under consideration. Because we do not want to be linked to a particular special shape for this region, we shall use periodic boundary conditions with an $L \times L \times L$ unit cube, i.e. we shall require that

$$\vec{u}_{\vec{l},\sigma}(\vec{r}) = \vec{u}_{\vec{l},\sigma}(\vec{r} + n_x L\vec{i}_x + n_y L\vec{i}_y + n_z L\vec{i}_z), \quad \text{for all integers } n_x, n_y, n_z. \tag{19}$$

Later, we shall take $L \to \infty$, to make our unit cube encompass all of space.

For finite L, consider

$$\vec{u}_{\vec{l},\sigma}(\vec{r}) = \frac{1}{\sqrt{L^3}}\, e^{j\vec{k}_{\vec{l}}\cdot\vec{r}}\vec{e}_{\vec{l},\sigma}, \tag{20}$$

where $\vec{k}_{\vec{l}}$ and $\vec{e}_{\vec{l},\sigma}$ are 3D real-valued vectors with the latter having unit length. Plugging this expression into the Helmholtz equation gives,

$$[-\vec{k}_{\vec{l}}\cdot\vec{k}_{\vec{l}} + \omega_{\vec{l}}^2/c^2]\frac{1}{\sqrt{L^3}}\, e^{j\vec{k}_{\vec{l}}\cdot\vec{r}}\vec{e}_{\vec{l},\sigma} = \vec{0}, \tag{21}$$

[2]In particular, our choice of a negative proportionality constant that is independent of the polarization index will be justified when we specialize to plane-wave solutions with periodic boundary conditions.

[3]As a result, we know that $q_{\vec{l},\sigma}(t) = q_{\vec{l},\sigma}e^{-j\omega_{\vec{l}}t}$ is the solution to this equation, with $q_{\vec{l},\sigma}$ being the (complex-valued) initial condition at $t = 0$.

for which the dispersion relation

$$\vec{k}_{\vec{l}} \cdot \vec{k}_{\vec{l}} = \omega_{\vec{l}}^2/c^2 \tag{22}$$

must hold if there are to be non-trivial solutions. Imposing the periodic boundary conditions then forces

$$\vec{k}_{\vec{l}} = (2\pi/L) \left[\begin{array}{ccc} l_x & l_y & l_z \end{array} \right]^T, \tag{23}$$

where $\{l_x, l_y, l_z\}$ are integers, at least one of which is non-zero, so that we can have $(2\pi/L)^2(l_x^2 + l_y^2 + l_z^2) = \omega_{\vec{l}}^2/c^2$.

To determine the constraints on the unit vector $\vec{e}_{\vec{l},\sigma}$, we return to the Coulomb gauge condition, from which we find

$$\nabla \cdot [q_{\vec{l},\sigma}(t)\vec{u}_{\vec{l},\sigma}(\vec{r})] = q_{\vec{l},\sigma}(t)[\nabla \cdot \vec{u}_{\vec{l},\sigma}(\vec{r})] = 0, \tag{24}$$

for the mode indexed by $(\vec{l}, \sigma)$. With our assumed form for $\vec{u}_{\vec{l},\sigma}(\vec{r})$ we then get the transversality condition,

$$j\vec{k}_{\vec{l}} \cdot \vec{e}_{\vec{l},\sigma} = 0, \tag{25}$$

i.e., the unit vector $\vec{e}_{\vec{l},\sigma}$ is orthogonal to $\vec{k}_{\vec{l}}$. Thus, for each $\vec{l}$ we only need two orthogonal $\vec{e}_{\vec{l},\sigma}$ vectors, viz., the two 3D unit vectors—indexed by $\sigma = 0, 1$—that are orthogonal to $\vec{k}_{\vec{l}}$.

What we have just derived are the plane-wave modes of the classical electromagnetic field:

$$\vec{A}(\vec{r}, t) = \frac{1}{2\sqrt{\epsilon_0 L^3}} \sum_{\vec{l},\sigma} q_{\vec{l},\sigma} e^{-j(\omega_{\vec{l}} t - \vec{k}_{\vec{l}} \cdot \vec{r})} \vec{e}_{\vec{l},\sigma} + \text{cc}, \tag{26}$$

where $k_{\vec{l}}$ obeys (22) and (23), and $\vec{e}_{\vec{l},\sigma}$ obeys (25). These are plane-wave modes. In particular, that they are waves propagating at speed c in the direction of $\vec{k}_{\vec{l}}$ can be seen by rewriting $\omega_{\vec{l}} t - \vec{k}_{\vec{l}} \cdot \vec{r}$ as $\omega_{\vec{l}}(t - \vec{i}_{\vec{l}} \cdot \vec{r}/c)$, where $\vec{i}_{\vec{l}} \equiv \vec{k}_{\vec{l}}/|\vec{k}_{\vec{l}}|$ and $|\vec{k}_{\vec{l}}| = \omega_{\vec{l}}/c$. That they are plane waves follows because $\vec{u}_{\vec{k},\sigma}$ is constant in planes perpendicular to $\vec{k}_{\vec{l}}$. We also note that the spatial mode functions are orthogonal, on the $L \times L \times L$ unit cube, because

$$\int_{L\times L\times L} \mathrm{d}^3\vec{r}\, \vec{u}_{\vec{l},\sigma}(\vec{r}) \cdot \vec{u}^{\,*}_{\vec{l}',\sigma'}(\vec{r}) = \frac{1}{L^3} \int_{L\times L\times L} \mathrm{d}^3\vec{r}\, e^{j(\vec{k}_{\vec{l}} - \vec{k}_{\vec{l}'})\cdot\vec{r}} \vec{e}_{\vec{l},\sigma} \cdot \vec{e}_{\vec{l}',\sigma'} \tag{27}$$

$$= \begin{cases} 0, & \text{for } l \neq l' \text{ or } \sigma \neq \sigma' \\ 1, & \text{for } l = l' \text{ and } \sigma = \sigma', \end{cases} \tag{28}$$

where $\vec{l} \neq \vec{l}'$ orthogonality follows from (23), and $\sigma \neq \sigma'$ orthogonality follows from (25).

The plane-wave modal decomposition of the vector potential gives rise, via (3), to the following plane-wave decompositions for the electric and magnetic fields,

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} = \sum_{\vec{l},\sigma} \frac{j\omega_{\vec{l}}}{2\sqrt{\epsilon_0 L^3}}\, q_{\vec{l},\sigma}\, e^{-j(\omega_{\vec{l}}t - \vec{k}_{\vec{l}}\cdot\vec{r})}\, \vec{e}_{\vec{l},\sigma} + \text{cc} \tag{29}$$

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t) = \sum_{\vec{l},\sigma} \frac{jc}{2\sqrt{\mu_0 L^3}}\, q_{\vec{l},\sigma}\, e^{-j(\omega_{\vec{l}}t - \vec{k}_{\vec{l}}\cdot\vec{r})}\, \vec{k}_{\vec{l}}\times\vec{e}_{\vec{l},\sigma} + \text{cc}. \tag{30}$$

It will be convenient—as it was for the harmonic oscillator—to go to replace the $\{q_{\vec{l},\sigma}\}$ with a dimensionless reformulation $\{a_{\vec{l},\sigma}\}$ defined by

$$a_{\vec{l},\sigma}(t) = \sqrt{\frac{\omega_{\vec{l}}}{2\hbar}}\, q_{\vec{l},\sigma}(t), \tag{31}$$

which reduces the preceding electric and magnetic fields to

$$\vec{E}(\vec{r},t) = \sum_{\vec{l},\sigma} j\sqrt{\frac{\hbar\omega_{\vec{l}}}{2\epsilon_0 L^3}}\, a_{\vec{l},\sigma}\, e^{-j(\omega_{\vec{l}}t - \vec{k}_{\vec{l}}\cdot\vec{r})}\, \vec{e}_{\vec{l},\sigma} + \text{cc} \tag{32}$$

$$\vec{H}(\vec{r},t) = \sum_{\vec{l},\sigma} j\sqrt{\frac{\hbar c^2}{2\mu_0\omega_{\vec{l}} L^3}}\, a_{\vec{l},\sigma}\, e^{-j(\omega_{\vec{l}}t - \vec{k}_{\vec{l}}\cdot\vec{r})}\, \vec{k}_{\vec{l}}\times\vec{e}_{\vec{l},\sigma} + \text{cc}. \tag{33}$$

The first term on the right in (32) is the positive-frequency electric field, denoted $\vec{E}^{(+)}(\vec{r},t)$, and likewise the first term on the right in (33) is the positive-frequency magnetic field, denoted $\vec{H}^{(+)}(\vec{r},t)$. The second terms in these equations are then the negative-frequency fields, denoted $\vec{E}^{(-)}(\vec{r},t)$ and $\vec{H}^{(-)}(\vec{r},t)$. In classical electromagnetism, the negative-frequency fields are the conjugates of the positive-frequency fields. When we quantize the electromagnetic field, $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ will become Hermitian Hilbert-space operators for the electric and magnetic fields. The positive-frequency field operators, in this case, are then non-Hermitian, whose adjoints are the associated negative-frequency field operators.

Stored Energy and the Hamiltonian

Our last step—in our treatment of classical electromagnetic waves in a source-free region of free space—is to use the expressions obtained at the end of the previous subsection to evaluate the stored energy in (Hamiltonian for) the $L \times L \times L$ unit cube. We have that

$$H = \int_{L\times L\times L} \mathrm{d}^3\vec{r}\left[\frac{1}{2}\epsilon_0\vec{E}(\vec{r},t)\cdot\vec{E}(\vec{r}t) + \frac{1}{2}\mu_0\vec{H}(\vec{r},t)\cdot\vec{H}(\vec{r},t)\right] \quad (34)$$

$$= \sum_{\vec{l},\sigma}\left[\frac{1}{2}\epsilon_0\frac{\hbar\bar{\omega}_{\vec{l}}}{\epsilon_0}a^*_{\vec{l},\sigma}a_{\vec{l},\sigma} + \frac{1}{2}\mu_0\frac{\hbar c^2}{\mu_0\omega_{\vec{l}}}a^*_{\vec{l},\sigma}a_{\vec{l},\sigma}\vec{k}_{\vec{l}}\cdot\vec{k}_{\vec{l}}\right] = \sum_{\vec{l},\sigma}\hbar\omega_{\vec{l}}a^*_{\vec{l},\sigma}a_{\vec{l},\sigma}. \quad (35)$$

The last equality is especially pleasing, insofar as our imminent jump to quantized electromagnetic fields is concerned. It shows that the frequency-$\omega_{\vec{l}}$ plane-wave mode with dimensionless coefficient $a_{\vec{l},\sigma}$—whose temporal dependence has already been shown to be simple harmonic motion—contributes an energy term $\hbar\omega_{\vec{l}}a^*_{\vec{l},\sigma}a_{\vec{l},\sigma}$ to the Hamiltonian, exactly as we saw earlier this semester for our dimensionless reformulation of the classical harmonic oscillator associated with the LC circuit.

Quantum Electromagnetic Fields in Free Space

We are now ready to quantize the electromagnetic field in a source-free region of free space using the plane-wave modes for periodic boundary conditions on an $L \times L \times L$ unit cube.

The Quantum Field Operators

Because the classical plane-wave modes behave like a set of harmonic oscillators, each one gets quantized exactly as we did for the single harmonic oscillator associated with the LC circuit. In particular we write

$$\hat{\vec{E}}(\vec{r},t) = \sum_{\vec{l},\sigma} j\sqrt{\frac{\hbar\omega_{\vec{l}}}{2\epsilon_0 L^3}}\,\hat{a}_{\vec{l},\sigma}\,e^{-j(\omega_{\vec{l}}t-\vec{k}_{\vec{l}}\cdot\vec{r})}\,\vec{e}_{\vec{l},\sigma} + \mathrm{hc} \quad (36)$$

$$\hat{\vec{H}}(\vec{r},t) = \sum_{\vec{l},\sigma} j\sqrt{\frac{\hbar c^2}{2\mu_0\omega_{\vec{l}}L^3}}\,\hat{a}_{\vec{l},\sigma}\,e^{-j(\omega_{\vec{l}}t-\vec{k}_{\vec{l}}\cdot\vec{r})}\,\vec{k}_{\vec{l}}\times\vec{e}_{\vec{l},\sigma} + \mathrm{hc}, \quad (37)$$

for the electric and magnetic field operators, where hc denotes Hermitian conjugate (adjoint). The first term on the right in (36) is the positive-frequency electric field operator, $\hat{\vec{E}}^{(+)}(\vec{r},t)$, and likewise the first term on the right in (37) is the positive-frequency magnetic field operator, $\hat{\vec{H}}^{(+)}(\vec{r},t)$. The second terms in these equations are then the negative-frequency field operators,

$$\hat{\vec{E}}^{(-)}(\vec{r},t) = \left[\hat{\vec{E}}^{(+)}(\vec{r},t)\right]^\dagger \quad \text{and} \quad \vec{H}^{(-)}(\vec{r},t) = \left[\hat{\vec{H}}^{(+)}(\vec{r},t)\right]^\dagger. \quad (38)$$

The operators $\{\hat{a}_{\vec{l},\sigma}\}$ that appear in $\hat{\vec{E}}^{(+)}(\vec{r},t)$ and $\hat{\vec{H}}^{(+)}(\vec{r},t)$ are modal photon-annihilation operators, viz., they have the canonical commutators,

$$\left[\hat{a}_{\vec{l},\sigma}, \hat{a}^{\dagger}_{\vec{l}',\sigma'}\right] = \delta_{\vec{l}\vec{l}'}\,\delta_{\sigma\sigma'} = \begin{cases} 1, & \text{for } \vec{l} = \vec{l}' \text{ and } \sigma = \sigma' \\ 0, & \text{for } \vec{l} \neq \vec{l}' \text{ or } \sigma \neq \sigma', \end{cases} \tag{39}$$

where the $\{\hat{a}^{\dagger}_{\vec{l},\sigma}\}$, which appear in $\hat{\vec{E}}^{(-)}(\vec{r},t)$ and $\hat{\vec{H}}^{(-)}(\vec{r},t)$, are the associated modal photon-creation operators. The Hamiltonian for the field is then found to be the sum of the harmonic-oscillator Hamiltonians—the $\{\hat{H}_{\vec{l},\sigma}\}$—for its individual plane-wave modes, i.e.,

$$\hat{H} = \sum_{\vec{l},\sigma} \hat{H}_{\vec{l},\sigma} = \sum_{\vec{l},\sigma} \hbar\omega_{\vec{l}}\left[\hat{a}^{\dagger}_{\vec{l},\sigma}\hat{a}_{\vec{l},\sigma} + \frac{1}{2}\right], \tag{40}$$

where the $\hbar\omega_{\vec{l}}/2$ term in the $\hat{a}_{\vec{l},\sigma}$ mode's Hamiltonian is its zero-point energy, which, as we are well aware, is responsible for the zero-point fluctuations in its quadrature components, $\text{Re}(\hat{a}_{\vec{l},\sigma})$ and $\text{Im}(\hat{a}_{\vec{l},\sigma})$.

Multi-Mode Number States and Coherent States

Because the quantum electromagnetic field consists of a collection of orthogonal harmonic-oscillator modes whose annihilation and creation operators obey canonical commutation relations, it is easy to build up multi-mode number states and multi-mode coherent states for the quantum field operators by taking tensor products of single-mode (harmonic-oscillator) quantum states.

Let's first develop the multi-mode number states. We define

$$\hat{N}_{\vec{l},\sigma} \equiv \hat{a}^{\dagger}_{\vec{l},\sigma}\hat{a}_{\vec{l},\sigma}, \tag{41}$$

to be the number operator for the $\hat{a}_{\vec{l},\sigma}$ mode, and let $\{|n_{\vec{l},\sigma}\rangle_{\vec{l},\sigma} : n_{\vec{l},\sigma} = 0, 1, 2, \ldots,\}$ denote its number states, viz.,

$$\hat{N}_{\vec{l},\sigma}|n_{\vec{l},\sigma}\rangle_{\vec{l},\sigma} = n_{\vec{l},\sigma}|n_{\vec{l},\sigma}\rangle_{\vec{l},\sigma}. \tag{42}$$

The multi-mode number state, $|\mathbf{n}\rangle$ is defined by the tensor-product construction,

$$|\mathbf{n}\rangle = \otimes_{\vec{l},\sigma}|n_{\vec{l},\sigma}\rangle_{\vec{l},\sigma}. \tag{43}$$

It is an eigenket of the total photon number, $\hat{N} \equiv \sum_{\vec{l},\sigma}\hat{N}_{\vec{l},\sigma}$, because,

$$\hat{N}|\mathbf{n}\rangle = \sum_{\vec{l},\sigma}\hat{N}_{\vec{l},\sigma}|\mathbf{n}\rangle = \sum_{\vec{l},\sigma}\left[(\otimes_{(\vec{l}',\sigma')\neq(\vec{l},\sigma)}|n_{\vec{l}',\sigma'}\rangle_{\vec{l}',\sigma'}) \otimes \hat{N}_{\vec{l},\sigma}|n_{\vec{l},\sigma}\rangle_{\vec{l},\sigma}\right] \tag{44}$$

$$= \sum_{\vec{l},\sigma} n_{\vec{l},\sigma}(\otimes_{\vec{l}',\sigma'}|n_{\vec{l}',\sigma'}\rangle_{\vec{l}',\sigma'}) = \left(\sum_{\vec{l},\sigma} n_{\vec{l},\sigma}\right)|\mathbf{n}\rangle. \tag{45}$$

A similar calculation, which you should perform, shows that $|\mathbf{n}\rangle$ is also an eigenket of the total Hamiltonian, viz.,

$$\hat{H}|\mathbf{n}\rangle = \left(\sum_{\vec{l},\sigma} \hbar\omega_{\vec{l}}(n_{\vec{l},\sigma} + 1/2)\right)|\mathbf{n}\rangle. \tag{46}$$

Now let's use the modal coherent states, $\{|\alpha_{\vec{l},\sigma}\rangle_{\vec{l},\sigma}\}$, which obey

$$\hat{a}_{\vec{l},\sigma}|\alpha_{\vec{l},\sigma}\rangle_{\vec{l},\sigma} = \alpha_{\vec{l},\sigma}|\alpha_{\vec{l},\sigma}\rangle_{\vec{l},\sigma}, \tag{47}$$

to build up the multi-mode coherent states via the tensor product construction,

$$|\boldsymbol{\alpha}\rangle \equiv \otimes_{\vec{l},\sigma}|\alpha_{\vec{l},\sigma}\rangle_{\vec{l},\sigma}. \tag{48}$$

We then have that the classical positive-frequency field associated with this multi-mode coherent state is

$$\vec{E}^{(+)}(\vec{r},t) \equiv \langle\boldsymbol{\alpha}|\hat{\vec{E}}^{(+)}(\vec{r},t)|\boldsymbol{\alpha}\rangle = \sum_{\vec{l},\sigma} j\sqrt{\frac{\hbar\omega_{\vec{l}}}{2\epsilon_0 L^3}}\,\alpha_{\vec{l},\sigma}\,e^{-j(\omega_{\vec{l}}t - \vec{k}_{\vec{l}}\cdot\vec{r})}\,\vec{e}_{\vec{l},\sigma}, \tag{49}$$

from which it follows that

$$\hat{\vec{E}}^{(+)}(\vec{r},t)|\boldsymbol{\alpha}\rangle = \vec{E}^{(+)}(\vec{r},t)|\boldsymbol{\alpha}\rangle, \tag{50}$$

as can easily be verified. Because of this relationship, we will use the notation $|\vec{E}^{(+)}(\vec{r},t)\rangle$ for the multi-mode coherent state $|\boldsymbol{\alpha}\rangle$ whose mean positive-frequency field is $\vec{E}^{(+)}(\vec{r},t)$, to emphasize that it is a positive-frequency field-operator eigenket whose spatio-temporal eigenfunction is $\vec{E}^{(+)}(\vec{r},t)$.

The state $|\vec{E}^{(+)}(\vec{r},t)\rangle$ has special coherence properties. Consider the following normally-ordered quantum correlation function,

$$G^{(N,M)}(\mathbf{v},\mathbf{r},\mathbf{t};\mathbf{w},\mathbf{r}',\mathbf{t}') \equiv \left\langle \left(\prod_{n=1}^{N} \hat{E}_{v_n}^{(-)}(\vec{r}_n,t_n)\right)\left(\prod_{m=1}^{M} \hat{E}_{w_m}^{(+)}(\vec{r}_m{}',t_m')\right)\right\rangle. \tag{51}$$

For the positive-frequency field operator: $\mathbf{v} = \{v_1, v_2, \ldots, v_N\}$, with $v_n = x, y,$ or z, specifies a set of N Cartesian-component selections; $\mathbf{r} = \{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N\}$ specifies a set of N spatial sampling points; and $\mathbf{t} = \{t_1, t_2, \ldots, t_N\}$ specifies a set of N sampling times. For the negative-frequency field operator: $\mathbf{w} = \{w_1, w_2, \ldots, w_M\}$, with $w_m = x, y,$ or z, specifies a set of M Cartesian-component selections; $\mathbf{r}' = \{\vec{r}_1{}', \vec{r}_2{}', \ldots, \vec{r}_M{}'\}$ specifies a set of M spatial sampling points; and $\mathbf{t}' = \{t_1', t_2', \ldots, t_M'\}$ specifies a set of M sampling times. When the field is in the multi-mode coherent state $|\vec{E}^{(+)}(\vec{r},t)\rangle$, we find—and you should verify—that

$$G^{(N,M)}(\mathbf{v},\mathbf{r},\mathbf{t};\mathbf{w},\mathbf{r}',\mathbf{t}') \equiv \left(\prod_{n=1}^{N} E_{v_n}^{(-)}(\vec{r}_n,t_n)\right)\left(\prod_{m=1}^{M} E_{w_m}^{(+)}(\vec{r}_m{}',t_m')\right), \tag{52}$$

for all choices of the $\{N, \mathbf{v}, \mathbf{r}, \mathbf{t}, M, \mathbf{w}, \mathbf{r}', \mathbf{t}'\}$, where $\vec{E}^{(-)}(\vec{r},t) = \left[\vec{E}^{(+)}(\vec{r},t)\right]^*$. Were $\hat{\vec{E}}^{(+)}(\vec{r},t)$ a classical, vector-valued random function of space and time, and $\hat{\vec{E}}^{(-)}(\vec{r},t)$ its complex conjugate, then the preceding correlation-function factorization would imply that this field was in fact deterministic and equal to $\vec{E}^{(+)}(\vec{r},t)$. This property provides strong justification for our saying that the multi-mode coherent state represents a classical electromagnetic field.

Prelude to Continuous-Time Photodetection

Although some quantum optical communication calculations require that we employ the full polarization, space, and time properties of the positive-frequency field operator, many others permit us to dispense with some of that generality. Thus, in preparation for next time's treatment of continuous-time photodetection, let us simplify our field operator notation by jettisoning some features of the full field-operator description that won't be necessary in the what follows. We make the following assumptions:

Assumption 1 Only one polarization is excited, so that scalar field operators will suffice, i.e., $\hat{\vec{E}}^{(+)}(\vec{r},t) \to \hat{E}^{(+)}(\vec{r},t)$.

Assumption 2 Only $+z$-going plane waves are excited, so transverse, i.e., (x,y), spatial characteristics and $-z$-going waves can be suppressed. Thus

$$\hat{E}^{(+)}(\vec{r},t) \to \hat{E}^{(+)}(z,t) = \sum_{l>0} j\sqrt{\frac{\hbar\omega_l}{2\epsilon_0 L^3}}\,\hat{a}_l e^{-j(\omega_l t - k_l z)}, \tag{53}$$

where $\omega_l/c = k_l = 2\pi l/L$.

Assumption 3 Only a narrow bandwidth about a center frequency ω_o is excited, so that we can treat all the excited modes as having the same photon energy, viz.,

$$\hat{E}^{(+)}(z,t) = \sum_{l>0} j\sqrt{\frac{\hbar\omega_o}{2\epsilon_0 L^3}}\,\hat{a}_l e^{-j(\omega_l t - k_l z)}, \quad \text{for } -T/2 \le t \le T/2, \tag{54}$$

where $T = L/c$.

Assumption 4 We will work with a photon-units baseband field operator in the $z = 0$ plane. This means we will replace $\hat{E}^{(+)}(z=0,t)$ with[4]

$$\hat{E}(t) = \sum_{n=-\infty}^{\infty} \frac{\hat{a}_n}{\sqrt{T}}\,e^{-j2\pi n/T}, \quad \text{for } -T/2 \le t \le T/2. \tag{55}$$

[4]By letting the sum over n range from $-\infty$ to ∞ we are including negative-frequency terms in this field operator. However, because we only excite positive-frequency terms in a narrow bandwidth about the center frequency, we will only make measurements on this narrowband region. Thus it is fair to use $\hat{E}(t)$ as given, i.e., the negative-frequency terms in $\hat{E}(t)$ will be neither excited nor measured in the development that will follow.

We see that $\hat{E}(t)$ is given by an operator-valued Fourier series from which we get the following commutator result,

$$\left[\hat{E}(t), \hat{E}^\dagger(u)\right] = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{[\hat{a}_n, \hat{a}_m^\dagger]}{T} e^{-j2\pi(nt-mu)/T} \quad (56)$$

$$= \sum_{n=-\infty}^{\infty} \frac{e^{-j2\pi n(t-u)/T}}{T} = \delta(t-u). \quad (57)$$

Assumption 5 We will let the quantization interval become $-\infty < t < \infty$. Now the Fourier series from our last assumption becomes a Fourier integral,

$$\hat{E}(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \hat{\mathcal{E}}(\omega) e^{-j\omega t} \quad \text{and} \quad \hat{\mathcal{E}}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \hat{E}(t) e^{j\omega t}, \quad (58)$$

with the following commutator relations,

$$\left[\hat{E}(t), \hat{E}^\dagger(u)\right] = \delta(t-u) \quad \text{and} \quad \left[\hat{\mathcal{E}}(\omega), \hat{\mathcal{E}}^\dagger(\omega')\right] = 2\pi\delta(\omega-\omega'). \quad (59)$$

The Road Ahead

In Lectures 18 and 19 we will describe, compare, and contrast the semiclassical and quantum theories of continuous-time photodetection.