MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

Problem Set No. 10 6.632 Electromagnetic Wave Theory
Spring Term 2003

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**Reading assignment**: Section 4.3, 4.4, 5.1 J. A. Kong, "*Electromagnetic Wave Theory*"

**Problem P10.1**

In Magnetic Resonance Imaging (MRI) study, it is very useful to define a rotating frame of reference which rotates about the $z$ axis at the Larmor frequency ($\omega_0 = \gamma B_0$, $\gamma$ is the gyromagnetic ratio). Consider the bulk magnetic moment $M_0\hat{z}$ placed in a DC magnetic field $B_0\hat{z}$. When a MRI transmitting coil generates a magnetic field of frequency $\omega_1$, effective total magnetic field can be described as

$$\overline{B} = B_0\hat{z} + B_1(\cos(\omega_1 t)\hat{x} - \sin(\omega_1 t)\hat{y})$$

And the rotating coodinate can be defined as

$$\begin{aligned} \hat{x}' &= \hat{x}\cos(\omega_1 t) - \hat{y}\sin(\omega_1 t) \\ \hat{y}' &= \hat{x}\sin(\omega_1 t) + \hat{y}\cos(\omega_1 t) \\ \hat{z}' &= \hat{z} \end{aligned}$$

(1) Show that when $\omega_1 = \omega_0 = \gamma B_0$ (called "on resonance"), magnetic moment $\overline{M}$ will respond to this $B_1$ field as a rotation about the $x'$ axis in the rotating frame. In other words, the effective $B_1$ field is $B_1\hat{x}'$.
(2) Also show that if $\omega_1 \neq \omega_0$ (called "off resonance"), the effective $B_1$ is the vector sum of $B_1\hat{x}'$ and $\Delta B_0\hat{z}'$, where $\Delta B_0 = (\omega_0 - \omega_1)/\gamma$.

**Problem P10.2**

(a) In MRI, the resonance frequency (Larmor frequency) $\omega_0$ of a spin particle is related to the magnetic field $B_0$ by gyromagnetic ratio, $\omega_0 = \gamma B_0$. $^1$H nucleus has two spin states. The energy of the photon needed to cause a transition between the two spin states of $^1$H nucleus in a 1.5T magnetic field is $4.23 \times 10^{-26}$ J. What is the gyromagnetic ration of $^1$H? (Note that the energy $E$ of a photon at frequency $\omega$ is $E = \hbar\omega$, where $\hbar = 6.63 \times 10^{-34}$ J s.)

(b) A sample contains two small distinct water locations where there is $^1$H spin density. In a uniform field, each of the $^1$H have the same Larmor frequency. However, if a linear gradient $G_x$ is superimposed on the main magnetic filed $B_0$, the Larmor frequency will depend on position along the $x$ axis. $\omega = \gamma(B_0 + xG_x) = \omega_0 + \gamma x G_x$. The MRI spectrum contains frequencies of $63.872\,\text{MHz}$ and $63.867\,\text{MHz}$ when $B_0$ is 1.5 T and $G_x = 1 \times 10^{-2}$ T/m. What are the locations of the water?

**Problem P10.3**

(a) Consider an array of two out-of-phase but equal amplitude $\hat{z}$-directed Hertzian dipoles as shown in Fig. 10.3. The wavelength is $\lambda$.

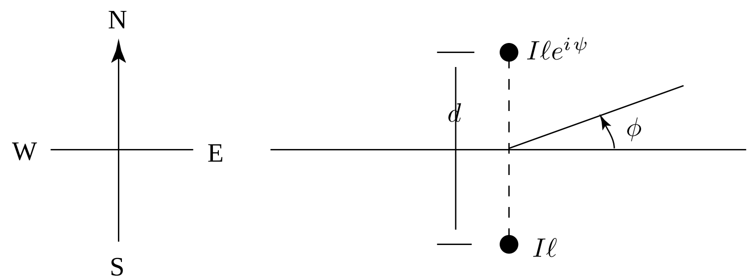


Fig. 10.3

Show that the array factor $|F(\phi)|$ may be expressed as

$$|F(\phi)| = \left|2\cos\left[\frac{kd}{2}\sin\phi - \frac{\psi}{2}\right]\right|$$

(b) A broadcast array of two vertical towers with equal current amplitude is to have a horizontal plane pattern such that
 (i) maximum field intensity is to the north ($\phi = 90°$)
 (ii) the only nulls are at $\phi = 225°$ and $\phi = 315°$.
 Specify the arrangement of the towers, their spacing and phasing.

**Problem P10.4**

An electric dipole antenna with dipole moment $I\ell$ is oriented in the $\hat{z}$ direction and is placed at the corner of a wall as shown in the figure. The ground and the wall are considered to be perfectly conducting and their areas are assumed to be infinite.

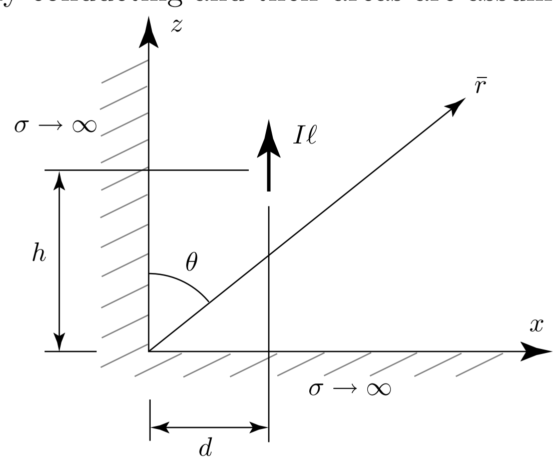


Fig. 10.4

(a) Find the three images of the dipole antenna. Indicate the coordinates and the orientations of the images.
(b) Explain why the radiation field from the dipole antenna is zero everywhere if $d = 0$.
(c) Let $h = 0$. The radiation pattern of the electric field $|E|$ is shown in Figure 10.4.c, where the maximum value $|E|_{\max}$ appears at $\theta = 90°$, and the nulls appear at $\theta = 0$ and $\theta_o$.

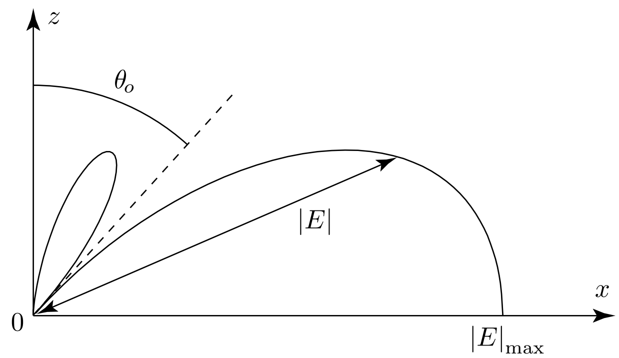


Fig. 10.4.c

(i) What is the value of $\theta_o$?
(ii) Find the distance $d$ in terms of wavelength $\lambda$.

(d) Let $h = 0$. Find the value(s) of $d$ in terms of wavelength $\lambda$ such that the radiated power along the $\hat{x}$ axis is zero.

(e) What is the field in region $z < 0$, and what is the field in region $x < 0$?

(f) Let $h = \lambda/2$ and $d = \lambda/4$. Using pattern multiplication technique, sketch the radiation pattern on the $xz$-plane.