Characterization of Left-Handed Materials

Massachusetts Institute of Technology
6.635 lecture notes

1 Introduction

1. How are they realized?
2. Why the denomination "Left-Handed"?
3. What are their properties?
4. Does it really work?

It has already been shown (see previous classes) that rings, or split-rings, can realize a negative permeability ($\mu < 0$) over a certain frequency band.

In addition to this, we need to realize a negative permittivity ($\epsilon < 0$).

It has also been shown (see previous classes) that:

- lossless:

$$\epsilon_{\text{metal}} = 1 - \frac{\omega_p^2}{\omega^2}, \qquad \text{where } \omega_p = \frac{ne^2}{\epsilon_0 m_e}$$

  (n: electron density, e: electron charge, m_e: effective mass of electrons).

- lossy:

$$\epsilon_{\text{metal}} = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)} .$$

A typical transmission curve looks like shown in Fig. 1.

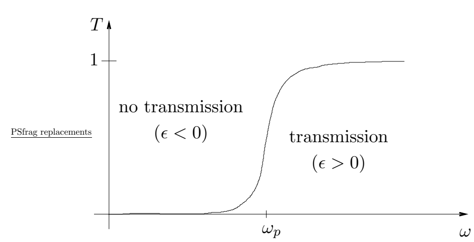

Figure 1: Transmission curve for a plasma-like medium.

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With these characteristics, $\epsilon < 0$ has been realized already at infrared frequencies (where metals behave like plasmas).

Problem: how to realize it at GHz frequencies?

Solution: by reducing n, the electron density.

One way of doing this is to confine the electrons in space. This can be achieved by an array of rods for example, as shown in Fig. 2.

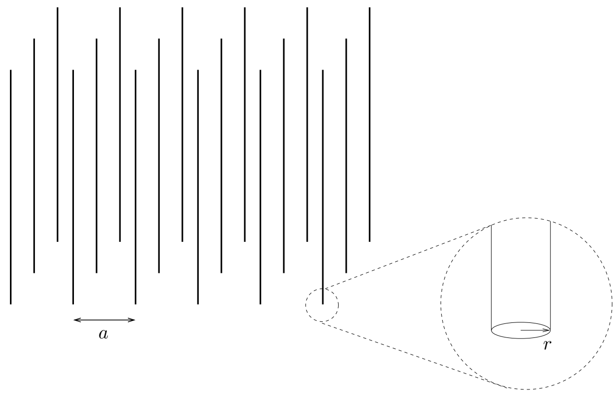


Figure 2: Array of rods confining the electrons in space.

Note: it is important that the wires are thin, so as to reduce the radiation interaction and allow penetration into the structure.

Effect of the wires: to reduce n to

$$n_{\text{eff}} = n\frac{\pi r^2}{a^2}\,. \tag{1}$$

Finally, note also that the rods have to be parallel to the electric field. This, plus the (known already) fact that rings have to be perpendicular to the magnetic field, gives an idea on how to realize physically LH metamaterials (see Fig. 3).

2 Why "left-handed"?

At this point, we have a metamaterial which can realize

$$\epsilon < 0,\ \mu < 0\,. \tag{2}$$

We shall now see what does it imply on the electromagnetic fields.

Let us write Maxwell's curl equations for plane wave solutions and time harmonic notations:

$$\bar{k} \times \bar{E}(\bar{r}) = \ \ \omega\mu\bar{H}(\bar{r})\,, \tag{3a}$$
$$\bar{k} \times \bar{H}(\bar{r}) = -\,\omega\epsilon\bar{E}(\bar{r})\,. \tag{3b}$$

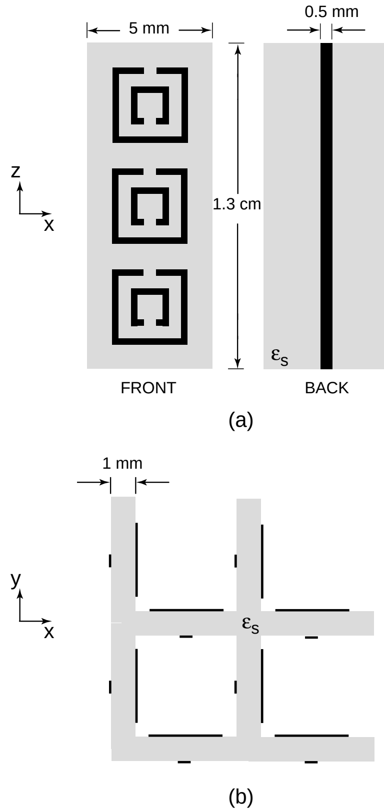


Figure 3: A realization for LH material.

In standard materials, Eq. (3) implies that the tryad $(\bar{E}, \bar{H}, \bar{k})$ forms a right-handed system. However, under Eq. (2), we will have:

$(\bar{E}, \bar{H}, \bar{k})$ form a left-handed (LH) tryad.

However, the time average Poynting power is still

$$< \bar{S}(\bar{r}) >= \frac{1}{2}\Re\{\bar{E}(\bar{r}) \times \bar{H}^{\star}(\bar{r})\} \tag{4}$$

and remains in the same direction so that we have the set up shown in Fig. 4.

Characteristics:

- $\bar{k}$ is in the phase velocity direction.
- phase velocity and energy flux are in opposite directions.

3 Properties of LH media

Some know characteristics are:

- Reversed Doppler effect (track the phase),
- Reversed Čerenkov radiation (cf. 6.632),

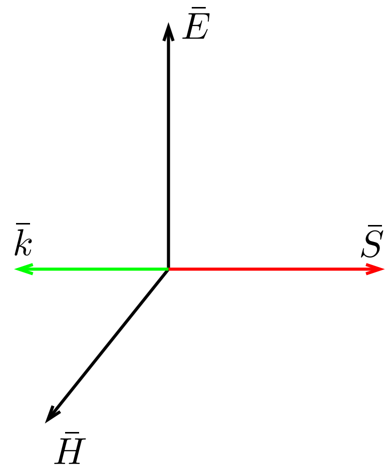


Figure 4: Electric field ($\bar{E}$), magnetic field ($\bar{H}$), wave-vector ($\bar{k}$) and Poynting power ($\bar{S}$) in an LH medium.

- Negative index of refraction.

This last item is very significant, and we shall spend some time discussing it.

The index of refraction of a medium is defined as

$$n = \sqrt{\epsilon_r \, \mu_r}\,, \tag{5}$$

or, writing explicitly the frequency dependence (cf. later),

$$n(\omega) = \sqrt{\epsilon_r(\omega)\, \mu_r(\omega)}\,. \tag{6}$$

For those frequencies inside the left-handed band (*i.e.* in the band where $\epsilon < 0$ and $\mu < 0$), we can write:

$$\epsilon(\omega) < 0 \quad \Rightarrow \quad \epsilon(\omega) = |\epsilon(\omega)|\, e^{i\pi}\,, \tag{7a}$$

$$\mu(\omega) < 0 \quad \Rightarrow \quad \mu(\omega) = |\mu(\omega)|\, e^{i\pi}\,, \tag{7b}$$

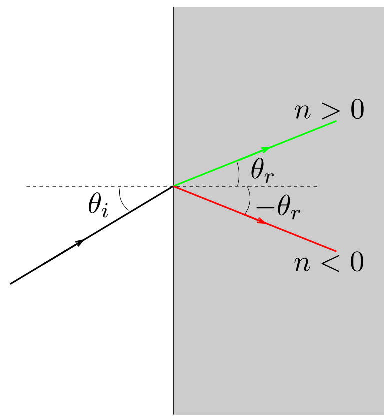
Eventually, we write n from Eq. (6):

$$n = \sqrt{|\epsilon(\omega)\mu(\omega)|\, e^{i\pi}} = -\sqrt{|\epsilon(\omega)\mu(\omega)|}\,. \tag{8}$$

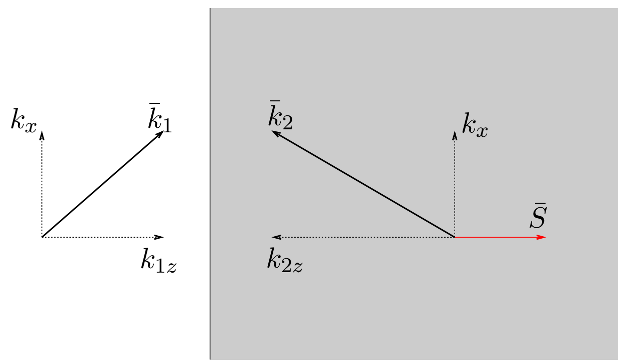
3.1 Reversed Snell's law

An important consequence of this fact is the reversal of Snell's law.

- Ray diagram:



- $\bar{k}$ diagram for an LH medium:



3.2 Energy

Traditionally, the energy is given by

$$W = \epsilon E^2 + \mu H^2 \,. \tag{9}$$

What happens if $\epsilon < 0$ and $\mu < 0$? Is $W < 0$?

Actually no, but this direct conclusion from Eq. (9) shows that this equation is not valid as is. In fact, these materials have to be modeled by frequency dispersive permittivity and permeability.

In that case, the relation of Eq. (9) becomes (from Poynting's theorem):

$$W = \frac{\partial(\epsilon\omega)}{\partial\omega}\, E^2 + \frac{\partial(\mu\omega)}{\partial\omega}\, H^2 \tag{10}$$

and we must have:

$$\frac{\partial(\epsilon\omega)}{\partial\omega} > 0\,, \tag{11a}$$

$$\frac{\partial(\mu\omega)}{\partial\omega} > 0\,. \tag{11b}$$

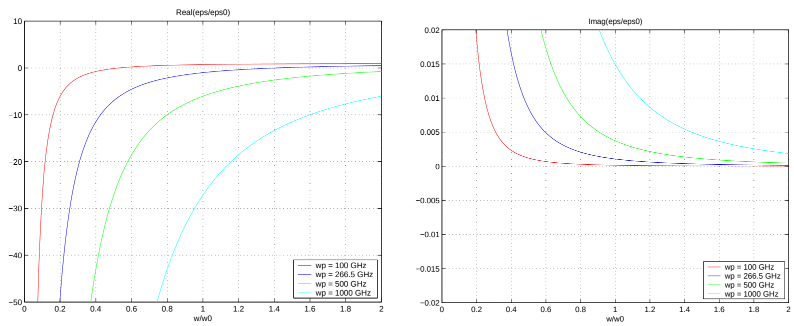
When LH materials are studies as bulk materials, two models are commonly used for the permittivity/permeability:

1. Drude model:

$$\epsilon_r = 1 - \frac{\omega_{ep}^2}{\omega(\omega + i\gamma_e)}, \tag{12a}$$

$$\mu_r = 1 - \frac{\omega_{mp}^2}{\omega(\omega + i\gamma_m)}, \tag{12b}$$

which is schematically represented in Fig. 5.



(a) $\Re(\epsilon)$. **(b)** $\Im(\epsilon)$.

Figure 5: Permittivity for various values of ω_{ep} in the Drude model ($f_0 = 30$ GHz, $\omega_{ep} = \omega_p$).

2. Resonant model:

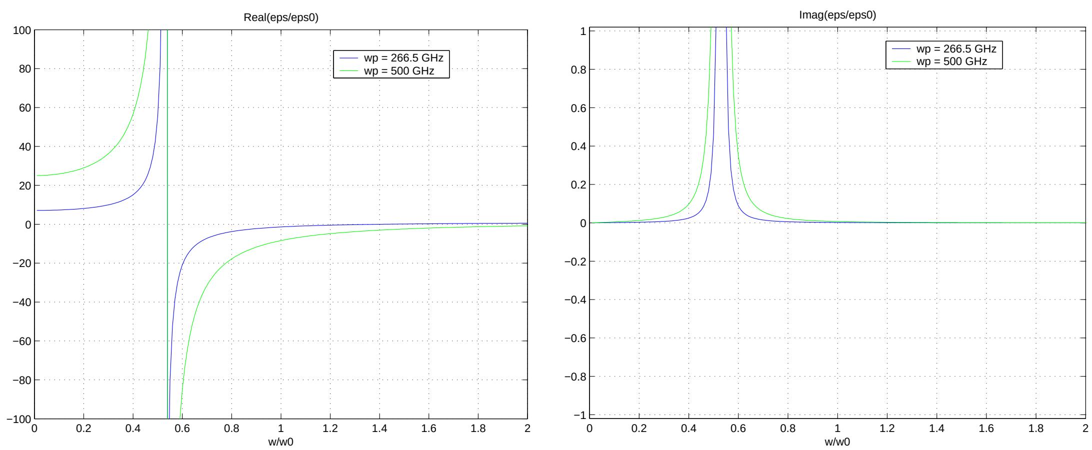
$$\epsilon_r = 1 - \frac{\omega_{ep}^2 - \omega_{eo}^2}{\omega^2 - \omega_{eo}^2 + i\gamma_e\omega}, \tag{13a}$$

$$\mu_r = 1 - \frac{\omega_{mp}^2 - \omega_{mo}^2}{\omega^2 - \omega_{mo}^2 + i\gamma_m\omega}, \tag{13b}$$

where $\omega_{(em)o}$ are the electric/magnetic resonant frequencies and $\omega_{(em)p}$ are the electric/magnetic plasma frequencies. An illustration of this model is given in Fig. 6.

3.3 Properties of an LHM slab

Let us consider the situation depicted in Fig. 7.



(a) $\Re(\epsilon)$.

(b) $\Im(\epsilon)$.

Figure 6: Permittivity for various values of ω_{ep} in the resonant model of Eq. (13a) ($f_0 = 30$ GHz, $\omega_{ep} = \omega_p$, $\omega_0 = 100GHz$).

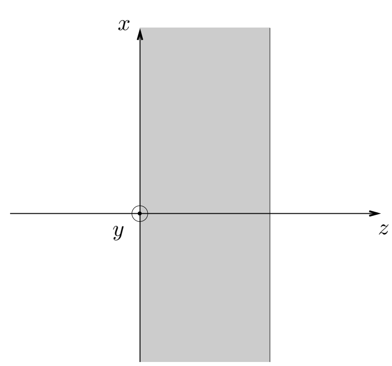
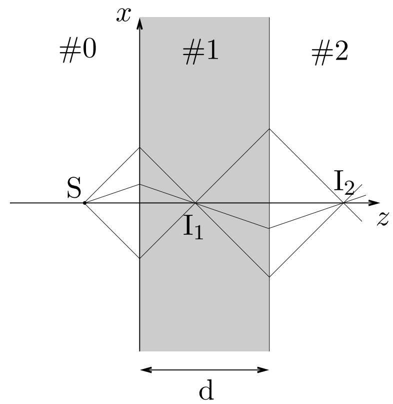


Figure 7: LH slab in free-space.

Let us consider the case for which $\epsilon_r = \mu_r = -1$ (working at the right frequency).

1. From simple ray diagrams, using the reversed Snell's law:



We see that is the source is close enough to the slab (distance$< d$), the slab will produce two images, one inside the slab and one outside.

The distance from source to the second image is

$$S - I_2 = 2d\,.$$

2. Rigorous calculation:

Let us consider a TE wave impinging on this slab. We write, for a single interface:

$$\bar{E}_0 = \hat{y} E_0\, e^{ik_x x - i\omega t}[e^{ik_z z} + r e^{-ik_z z}]\,, \tag{14a}$$
$$\bar{E}_1 = \hat{y} E_0\, e^{ik_x x - i\omega t}\, t e^{ik'_z z}\,. \tag{14b}$$

We need to match the boundary conditions and, for simplicity, we set the boundary to be at $z = 0$. We get:

- Tangential $\bar{E}$ field:

$$e^{ik_z z} + r e^{-ik_z z} = t e^{ik'_z z}\,.$$

- Tangential $\bar{H}$ field ($\bar{H} = \frac{1}{i\omega\mu}\nabla \times \bar{E}$):

$$e^{ik_z z} - r e^{-ik_z z} = \frac{\mu_0 k'_z}{\mu_1 k_z}\, t e^{ik'_z z}\,.$$

Upon solving, we get the reflection/transmission coefficient from free-space to the medium:

$$t = \frac{2\mu_1 k_z}{\mu_1 k_z + \mu_0 k'_z}\,, \tag{15a}$$
$$r = \frac{\mu_1 k_z - \mu_0 k'_z}{\mu_1 k_z + \mu_0 k'_z}\,. \tag{15b}$$

In a similar way, the reflection/transmission coefficients from the medium to free-space are:

$$t' = \frac{2\mu_0 k_z'}{\mu_1 k_z + \mu_0 k_z'}\,, \tag{16a}$$

$$r' = \frac{\mu_0 k_z' - \mu_1 k_z}{\mu_1 k_z + \mu_0 k_z'}\,. \tag{16b}$$

In order to obtain the field inside the slab, we shall compute the transmission coefficient as:

$$\begin{aligned} T &= te^{ik_z'd}\,t' + te^{ik_z'd}\,r'e^{ik_z'd}\,r'e^{ik_z'd}\,t' + tt'(r'^2)^2 e^{5ik_z'd} + \cdots \\ &= tt'e^{ik_z'd}\sum_{n=0}^{\infty}(r'^2)^n\,e^{2ink_z'd} = \frac{tt'e^{ik_z'd}}{1 - r'^2 e^{2ik_z'd}}\,. \end{aligned} \tag{17}$$

Using the previous expressions, we obtain

$$T = \frac{4\mu\mu_1 k_z k_z' e^{ik_z'd}}{(\mu_1 k_z + \mu_0 k_z')^2 - (\mu_0 k_z' - \mu_1 k_z)e^{2ik_z'd}}\,. \tag{18}$$

For the specific case of $\epsilon_r = \mu_r = -1$:

$$T = \frac{-4k_z k_z' e^{ik_z'd}}{(-k_z + k_z')^2 - (k_z' + k_z)e^{2ik_z'd}}\,. \tag{19}$$

At this point, we need to define k_z and k_z':

	Propagating waves ($k > k_\rho$)	Evanescent waves ($k < k_\rho$)
k_z	$k_z = \sqrt{k^2 - k_\rho^2}$	$k_z = i\sqrt{k_\rho^2 - k^2}$
k_z'	$k_z' = -\sqrt{k^2 - k_\rho^2}$	$k_z' = i\sqrt{k_\rho^2 - k^2}$
Relation	$k_z' = -k_z$	$k_z' = k_z$

Performing the calculation, we get:

- Propagating waves ($k_z' = -k_z$):

$$T = \frac{4k_z^2 e^{-ik_z d}}{4k_z^2} = e^{-ik_z d}\,.$$

- Evanescent waves ($k_z' = k_z$):

$$T = \frac{-4k_z^2 e^{ik_z d}}{-4k_z^2 e^{2ik_z d}} = e^{-ik_z d}\,.$$

Therefore, for all waves, we get:

$$T = e^{-ik_z d}. \tag{20}$$

Conclusions:

- Evanescent waves are amplified by the medium.
- Propagating waves are “backward waves”.
- Taking an infinite amount of those creates a source.
- Two images can be formed, like shown in the ray diagram of the previous subsection.

4 Does it really work?

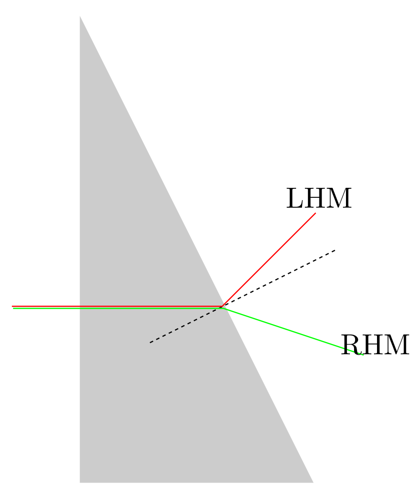
The theoretical predictions have been verified in experiments.

To date, essentially two experiments have been done:

1. Prism,
2. Gaussian beam.

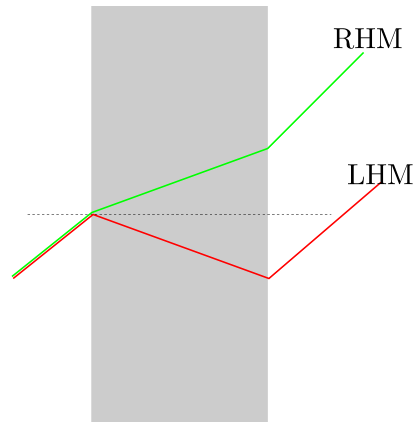
4.1 Prism experiment

First experiment (2001). Later reproduced.



4.2 Gaussian beam experiment

First publication was theoretical. Later experiments followed, but still need to be improved.


RHM
LHM