Dispertion relations in Left-Handed Materials

Massachusetts Institute of Technology
6.635 lecture notes

1 Introduction

We know already the following properties of LH media:

1. ϵ_r and μ_r are frequency dispersive.
2. ϵ_r and μ_r are negative over a similar frequency band.
3. The tryad $(\bar{E}, \bar{H}, \bar{k})$ is left-handed.
4. The index of refraction is negative.

From the past lectures, we know that these materials can be realized by a succession of wires and rods:

- Periodic arrangement of rods: realizes a plasma medium with negative ϵ_r over a certain frequency band. The model for the permittivity is:

$$\epsilon_r = 1 - \frac{\omega_{ep}^2}{\omega^2 + i\gamma_e\omega} \,. \tag{1}$$

- Periodic arrangement of rings (split-rings) realizes a resonant μ_r modeled as

$$\mu_r = 1 - \frac{F\omega_{mp}^2}{\omega^2 - \omega_{mo}^2 + i\gamma_m\omega} \,, \tag{2}$$

  where F is the fractional area of the unit cell occupied by the interior of the split-ring $(F < 1)$.

In the lossless case $(\gamma_e = \gamma_m = 0)$, we can rewrite these two relations as:

$$\epsilon_r = \frac{\omega^2 - \omega_{ep}^2}{\omega^2} \,, \tag{3a}$$

$$\mu_r = \frac{\omega^2 - \omega_0^2 - F\omega^2}{\omega^2 - \omega_0^2} = (1 - F)\,\frac{\omega^2 - \omega_b^2}{\omega^2 - \omega_0^2} \,, \tag{3b}$$

where $\omega_b = \omega_0/\sqrt{1-F} > \omega_0$. Therefore:

$$k^2 = \omega^2\epsilon\mu = \frac{1}{c}(1 - F)\frac{(\omega^2 - \omega_{ep}^2)(\omega^2 - \omega_b^2)}{\omega^2 - \omega_0^2} \,. \tag{4}$$

Upon identifying the regions where ϵ_r and μ_r change signs, we can immediately get the relation for k:

ω		ω_0		ω_b		ω_p	
ϵ_r	$-$		$-$		$-$		$+$
μ_r	$+$		$-$		$+$		$+$
k^2	$-$		$+$		$-$		$+$

The region $\omega \in [\omega_0, \omega_b]$, which also corresponds to $\epsilon_r < 0$ and $\mu_r < 0$, corresponds to k^2 positive, which means k real. Therefore, there is propagation in this band, but not in the adjacent ones.

It may still not be clear that k is negative, even if we write

$$k = \omega^2 \sqrt{\epsilon\mu} = \omega^2 \sqrt{\epsilon_0^2 \mu_0^2}\, \sqrt{\epsilon_r \mu_r} = k_0 n\,. \tag{5}$$

A demonstration of the fact that n is negative follows.

2 Argument on $n < 0$

2.1 Complex Poynting theorem

We shall first recall the derivation of the complex Poynting theorem and the signification of the various terms.

We start from Maxwell's curl equation

$$\nabla \times \bar{E} = \ \ i\omega \bar{B}\,, \tag{6a}$$

$$\nabla \times \bar{H} = -\, i\omega \bar{D} + \bar{J}\,. \tag{6b}$$

Upon multiplying Eq. (6a) by $\bar{H}^\star$ and substracting the complex conjugate of Eq. (6b) multuplied by $\bar{E}$ we get:

$$\begin{aligned}\bar{H}^\star \cdot \nabla \times \bar{E} - \bar{E} \cdot \nabla \times \bar{H}^\star &= \nabla \cdot (\bar{E} \times \bar{H}^\star) \\ &= i\omega \bar{B} \cdot \bar{H}^\star - i\omega \bar{D}^\star \cdot \bar{E} - \bar{J}^\star \cdot \bar{E} \\ &= i\omega[\bar{B} \cdot \bar{H}^\star - \bar{E} \cdot \bar{D}^\star] - \bar{E} \cdot \bar{J}^\star\,. \end{aligned} \tag{7}$$

Upon rewriting, we get:

$$-\bar{E} \cdot \bar{J}^\star = \nabla \cdot (\bar{E} \times \bar{H}^\star) + i\omega[\bar{E} \cdot \bar{D}^\star - \bar{B} \cdot \bar{H}^\star]\,. \tag{8}$$

On the right-hand side of the equation, the first terms corresponds to the divergence of Poyting power, which is therefore positive. The second term relates to the complex EM energy, and is therefore also positive. Consequently, the left-hand side term must also be positive, and actually corresponds to the power supplied by $\bar{J}$ to the volume.

We shall use this result hereafter.

2.2 1D wave equation

For the sake of simplification, let us work with a 1D problem. The wave equation

$$\nabla^2 \bar{E}(\bar{r}) + k^2 \bar{E}(\bar{r}) = -i\omega\mu \bar{J}(\bar{r})\,, \tag{9}$$

is rewritten with

$$\bar{E}(\bar{r}) = \hat{z}\, E(x)\,, \tag{10a}$$

$$\bar{J}(\bar{r}) = \hat{z}\, j_0 \delta(x - x_0)\,, \tag{10b}$$

to yield

$$\frac{\partial^2}{\partial x^2} E(x) + k^2 E(x) = -i\omega\mu j_0 \delta(x - x_0)\,. \tag{11}$$

The solution to this equation is

$$E(x) = \alpha\, e^{ik|x-x_0|}\,, \tag{12}$$

where α needs to be determined. From Eq. (12), we write:

1. First derivative:
$$\frac{\partial E(x)}{\partial x} = \alpha ik\, \frac{\partial}{\partial x^2}|x - x_0|\;\; e^{ik|x-x_0|}\,. \tag{13}$$

2. Second derivative:
$$\begin{aligned}\frac{\partial^2 E(x)}{\partial x^2} =& \alpha ik\, \frac{\partial^2}{\partial x^2}|x - x_0|\;\; e^{ik|x-x_0|} + \alpha(-k^2)(\frac{\partial}{\partial x^2}|x - x_0|)^2\, e^{ik|x-x_0|} \\ =& -\alpha k^2\, e^{ik|x-x_0|} + 2i\alpha k \delta(x - x_0)\,.\end{aligned} \tag{14}$$

Therefore:

$$\frac{\partial^2 E(x)}{\partial x^2} + k^2 E(x) = 2i\alpha k \delta(x - x_0) = 2i\alpha\, k_0\, n\, \delta(x - x_0)\,. \tag{15}$$

Comparing Eq. (11) to Eq. (15), we get

$$\alpha = -\frac{\omega\mu j_0}{2k_0\, n} = -\frac{j_0\,\eta_0}{2}\,\frac{\mu_r}{n}\,, \tag{16}$$

so that finally the solution is:

$$E(x) = -\frac{j_0\,\eta_0}{2}\,\frac{\mu_r}{n}\;\, e^{ik|x-x_0|}\,. \tag{17}$$

If we now compute the power supplied by the current $\bar{J}$ to the volume:

$$P = -\frac{1}{2}\int_V \bar{E}\cdot\bar{J}^\star\, dV = \frac{\eta_0\, j_0^2}{4}\,\frac{\mu_r}{n}\; > 0\,. \tag{18}$$

The source must, on average, do positive work on the field. Yet, in LH regime, we have $\mu_r < 0$ so that we must have $n < 0$ as well.

Finally, we can also write the $\bar{E}$ field as:

$$E(x,t) = -\frac{\eta}{2}\, j_0\, e^{i(k_0 n|x-x_0|-\omega t)}\,. \tag{19}$$

Thus, plane waves appear to propagate from $-\infty$ and $+\infty$ to the source, seemingly running backward in time. Yet, the work done on the field is positive so clearly the energy propagates outward from the source.

3 Dispersion relations

At this point, we know that $n < 0$ and $k < 0$. The difference between phase and group velocity can be directly seen on the dispersion relation diagram.

$$v_\phi = \frac{\omega}{k_z}\,, \tag{20a}$$

$$v_g = \left(\frac{\partial k_z}{\partial \omega}\right)^{-1}\,. \tag{20b}$$

- Free-space: $k = \omega\sqrt{\epsilon\mu}$ where $\epsilon = cte$ and $\mu = cte$.
- Metamaterial:

Let us take the following models

$$\epsilon_r = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma_e\omega}\,, \tag{21a}$$

$$\mu_r = 1 - \frac{\omega_{mp}^2 - \omega_{mo}^2}{\omega^2 - \omega_{mo}^2 + i\gamma_m\omega}\,, \tag{21b}$$

3.1 Lossless case $(\gamma_e = \gamma_m = 0)$, $\omega_{mp} = \omega_p$

We rewrite

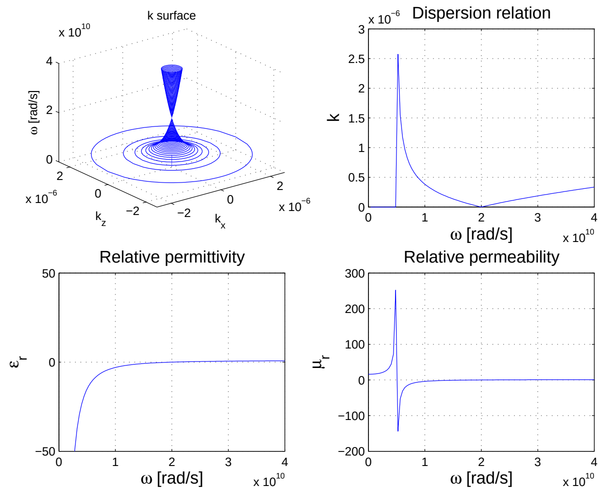
$$\epsilon_r = \frac{\omega^2 - \omega_p^2}{\omega^2}\,, \tag{22a}$$

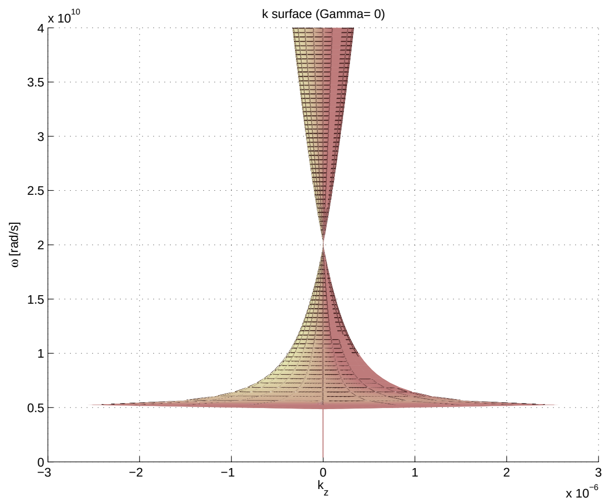
$$\mu_r = \frac{\omega^2 - \omega_{mp}^2}{\omega^2 - \omega_{mo}^2}\,, \tag{22b}$$

$$\tag{22c}$$

and plot the relations with

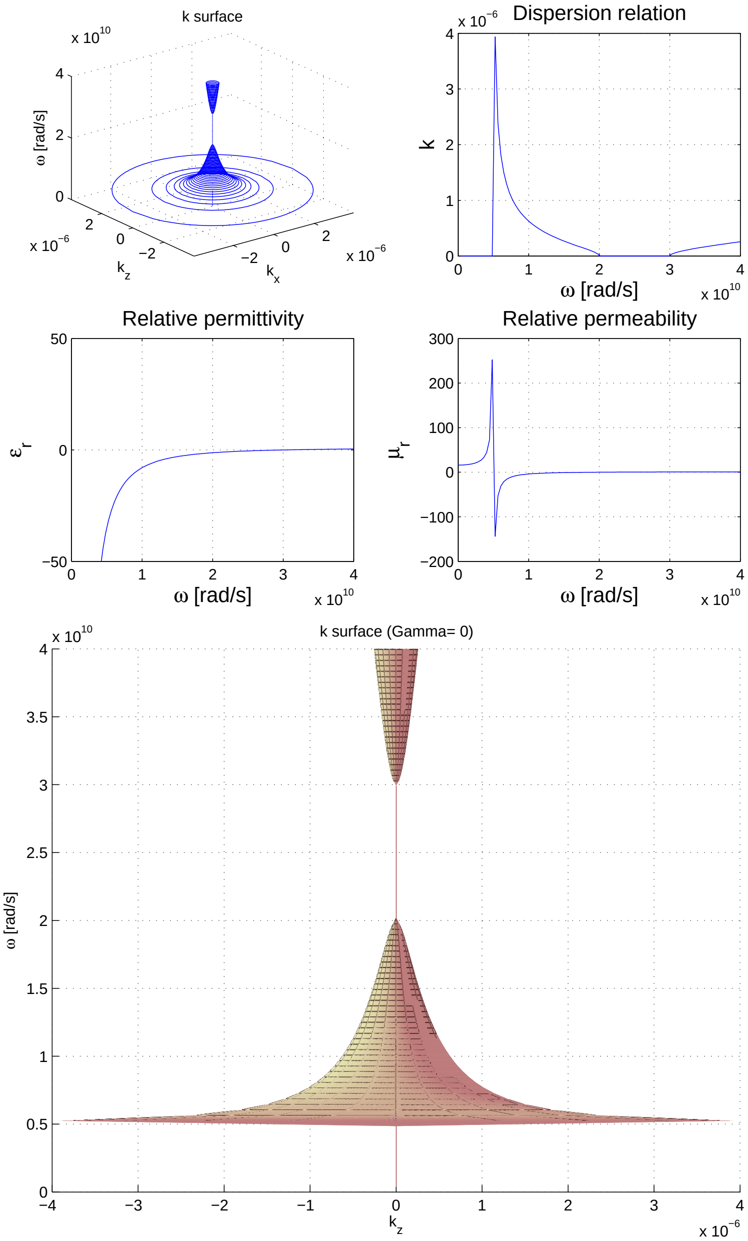
ω_p	$=$	20e9 rad/s
ω_{mp}	$=$	20e9 rad/s
ω_{mo}	$=$	5e9 rad/s
$\gamma_e = \gamma_m$	$=$	0



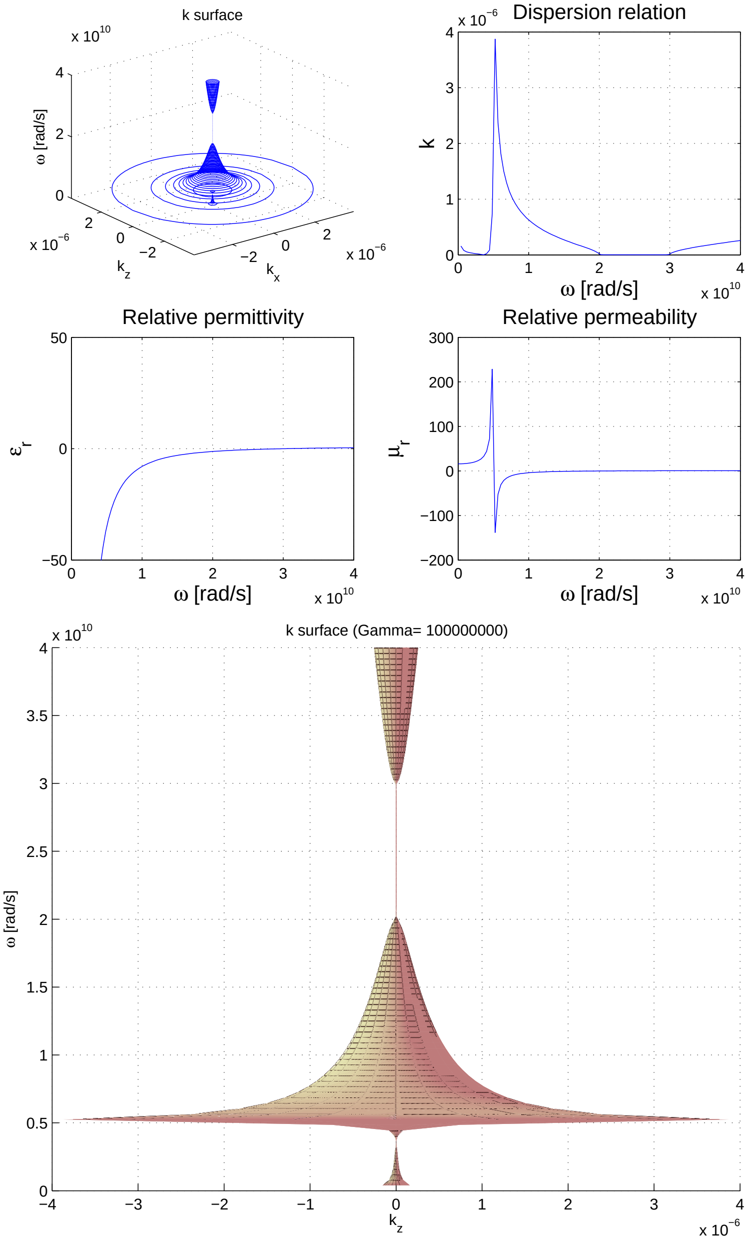


Other cases follow.

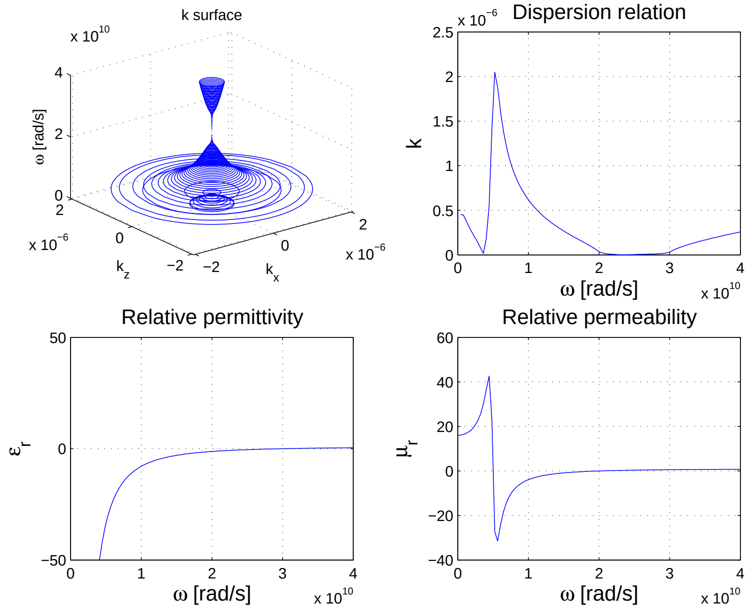
ω_p	$=$	30e9 rad/s
ω_{mp}	$=$	20e9 rad/s
ω_{mo}	$=$	5e9 rad/s
$\gamma_e = \gamma_m$	$=$	0 rad/s

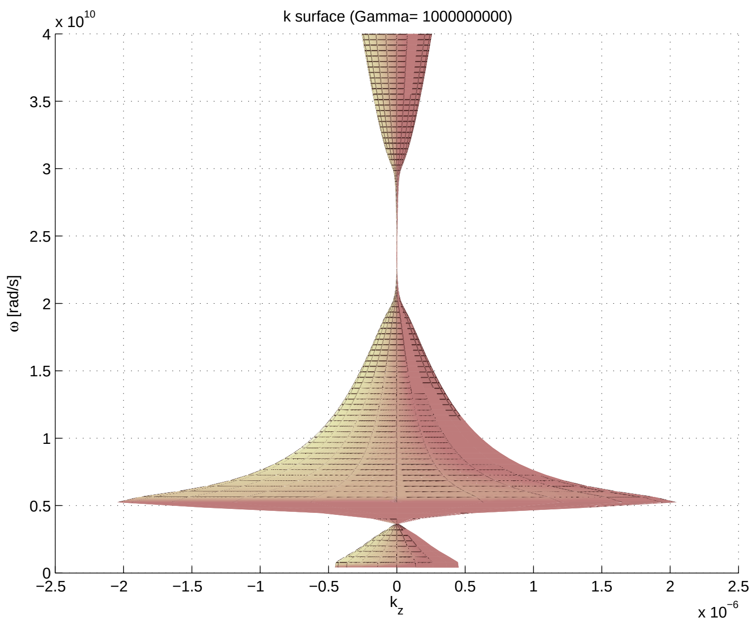


ω_p	$=$	30e9 rad/s
ω_{mp}	$=$	20e9 rad/s
ω_{mo}	$=$	5e9 rad/s
$\gamma_e = \gamma_m$	$=$	10e7 rad/s

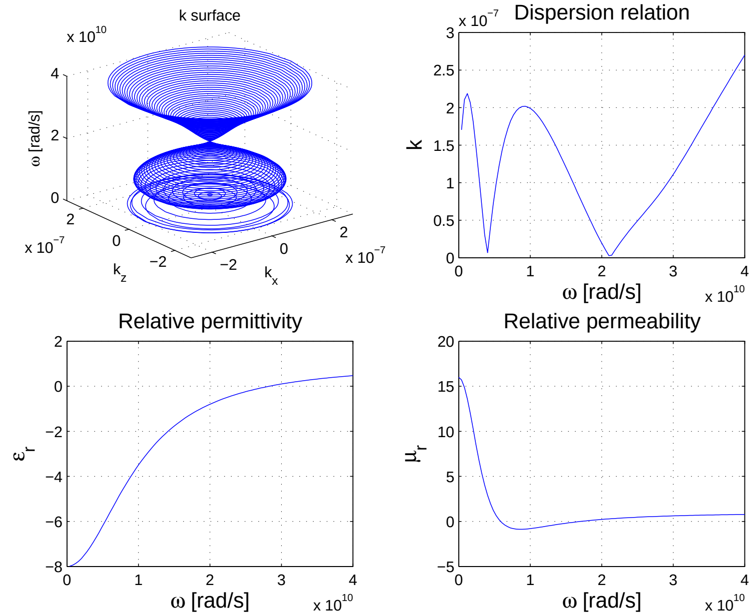


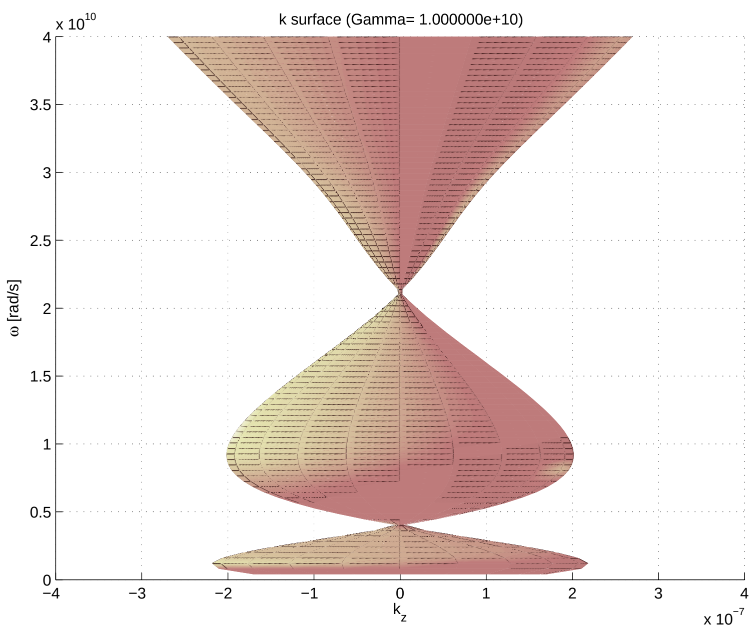
ω_p	=	30e9 rad/s
ω_{mp}	=	20e9 rad/s
ω_{mo}	=	5e9 rad/s
$\gamma_e = \gamma_m$	=	10e8 rad/s



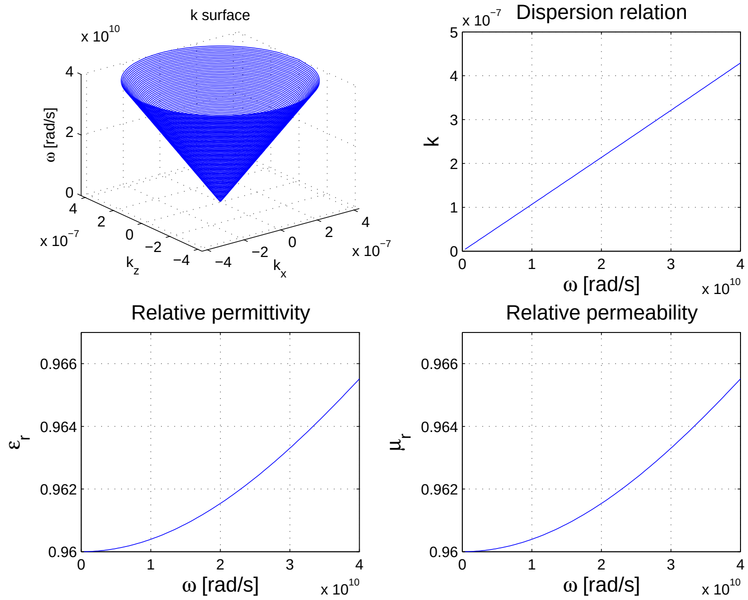


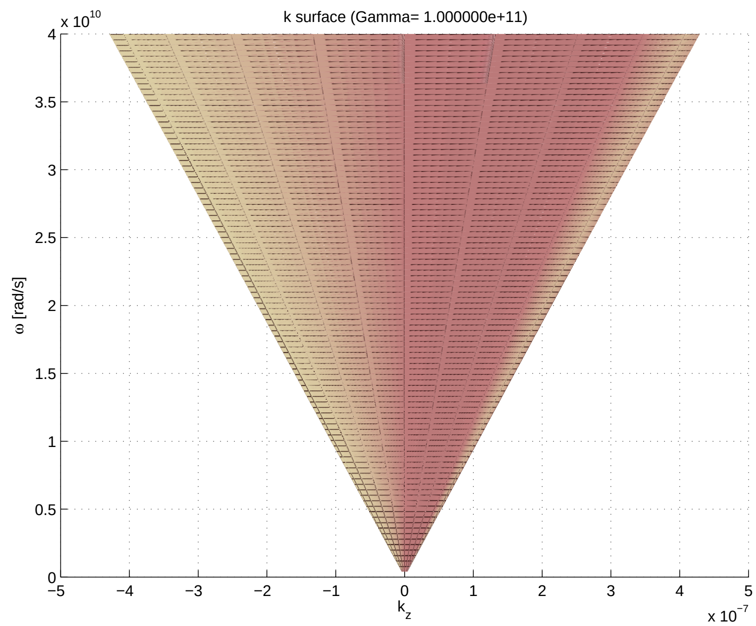
ω_p	$=$	30e9 rad/s
ω_{mp}	$=$	20e9 rad/s
ω_{mo}	$=$	5e9 rad/s
$\gamma_e = \gamma_m$	$=$	10e9 rad/s



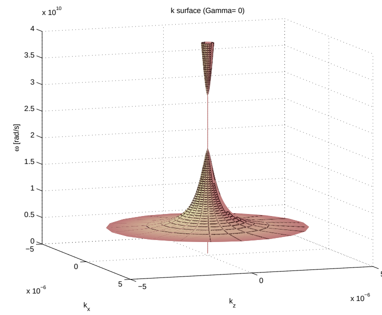


ω_p	$=$	30e9 rad/s
ω_{mp}	$=$	20e9 rad/s
ω_{mo}	$=$	5e9 rad/s
$\gamma_e = \gamma_m$	$=$	10e10 rad/s

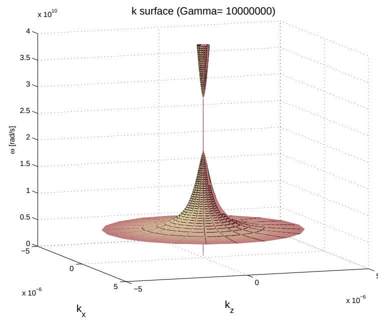




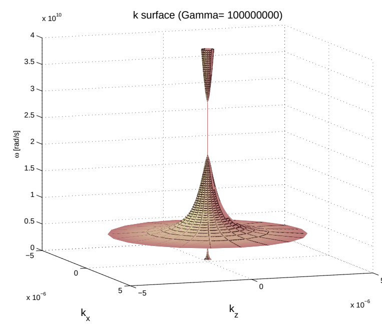
Plotting all the 3D curves on the same scale:



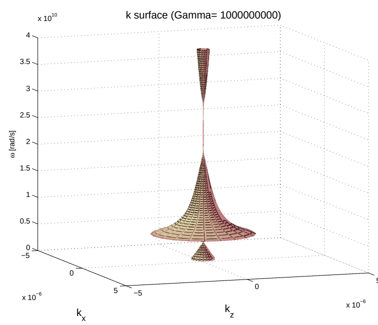
(a) $\gamma = 0$ rad/s.



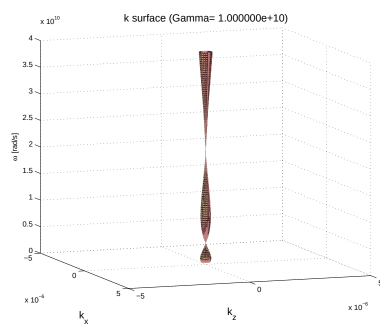
(b) $\gamma = 10e6$ rad/s.



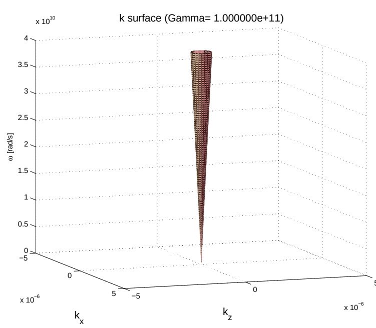
(c) $\gamma = 10e7$ rad/s.



(d) $\gamma = 10e8$ rad/s.



(e) $\gamma = 10e9$ rad/s.



(f) $\gamma = 10e10$ rad/s.

Figure 1: Dispersion relation (real k) for various values of losses γ. The models are: $\epsilon_r = 1 - \omega_p^2/(\omega^2 + i\gamma\omega)$. $\mu_r = 1 - (\omega_{mp}^2 - \omega_{mo}^2)/(\omega^2 - \omega_{mo}^2 + i\gamma\omega)$, where $\omega_p = 30e9$ rad/s, $\omega_{mp} = 20e9$ rad/s, $\omega_{mo} = 5e9$ rad/s.

For simplicity, we can study the lossy case for $\gamma_e = \gamma_m$ amd $\omega_{mo} = 0$ (although we don't really simulate the same medium, the fundamental behavior is similar, and simpler to carry out mathematically).

The model therefore reads:

$$\epsilon_r = \mu_r = \frac{\omega^2 - \omega_p^2 + i\gamma\omega}{\omega^2 + i\gamma\omega}\,. \tag{23}$$

We compute:

$$\begin{aligned}\sqrt{\epsilon_r \mu_r} &= \frac{\omega^2 - \omega_p^2 + i\gamma\omega}{\omega^2 + i\gamma\omega} \\ &= \frac{[\omega^2 - \omega_p^2 + i\gamma\omega]\,[\omega^2 - i\gamma\omega]}{\omega^4 + \gamma^2\omega^2} \\ &= \frac{\omega^2(\omega^2 - \omega_p^2) + \gamma^2\omega^2 + i\gamma\omega\omega_p^2}{\omega^4 + \gamma^2\omega^2}\,.\end{aligned} \tag{24a}$$

The real part is given by:

$$\Re\{\sqrt{\epsilon_r \mu_r}\} = \frac{\omega^2[\omega^2 - (\omega_p^2 - \gamma^2)]}{\omega^4 + \gamma^2\omega^2}\,. \tag{25}$$

Losses have the effect to lower the plasma frequency to

$$\omega_{p'} = \sqrt{\omega_p^2 - \gamma^2}\,. \tag{26}$$

In addition, we also see that if γ is very large, the plasma effect will completey dissapear (cf. dispersion relation for $\gamma = 10e10$ rad/s).