Time Domain Method of Moments

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6.635 lecture notes

1 Introduction

The Method of Moments (MoM) introduced in the previous lecture is widely used for solving integral equations in the frequency domain. Yet, some attempts have been made recently at the use of the MoM in the time domain. We shall briefly expose this approach here.

2 Time domain equations

The first step is of course to write Maxwell's equation and all other relations (constitutive relations and continuity) in time domain:

$$\nabla \times \bar{E}(\bar{r},t) = -\frac{\partial}{\partial t}\bar{B}(\bar{r},t) - \bar{M}(\bar{r},t)\,, \qquad \nabla \times \bar{H}(\bar{r},t) = \frac{\partial}{\partial t}\bar{D}(\bar{r},t) + \bar{J}(\bar{r},t)\,, \tag{1a}$$

$$\nabla \cdot \bar{B}(\bar{r},t) = m(\bar{r},t)\,, \qquad \nabla \cdot \bar{D}(\bar{r},t) = \rho(\bar{r},t)\,, \tag{1b}$$

$$\bar{D}(\bar{r},t) = \epsilon\bar{E}(\bar{r},t)\,, \qquad \bar{B}(\bar{r},t) = \mu\bar{H}(\bar{r},t)\,, \tag{1c}$$

$$\nabla \cdot \bar{J}(\bar{r},t) + \frac{\partial}{\partial t}\rho(\bar{r},t) = 0\,, \qquad \nabla \cdot \bar{M}(\bar{r},t) + \frac{\partial}{\partial t}m(\bar{r},t) = 0\,. \tag{1d}$$

For the time-domain MoM, it is easier to work with the potentials, and make use of the well-known "retarded potentials" theory. In view of doing this, we write the definition:

$$\bar{H}(\bar{r},t) = \frac{1}{\mu_0}\nabla \times \bar{A}(\bar{r},t) \tag{2a}$$

$$\bar{E}(\bar{r},t) = -\nabla\phi(\bar{r},t) - \frac{\partial}{\partial t}\bar{A}(\bar{r},t)\,. \tag{2b}$$

Both the vector potential $\bar{A}$ and the scalar potential ϕ satisfy the wave equation which, in time-domain domain, writes:

$$\nabla^2\bar{A}(\bar{r},t) - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\bar{A}(\bar{r},t) = -\mu_0\bar{J}(\bar{r},t)\,, \tag{3a}$$

$$\nabla^2\phi(\bar{r},t) - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\phi(\bar{r},t) = -\frac{\rho(\bar{r},t)}{\epsilon_0}\,. \tag{3b}$$

These potentials are linked by the time-domain Lorentz gauge:

$$\nabla \cdot \bar{A}(\bar{r},t) + \epsilon_0\mu_0\frac{\partial}{\partial t}\phi(\bar{r},t) = 0\,. \tag{4}$$

We can defined also a time-domain Green's function which satisfies the time-domain scalar equation:

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) g(\bar{r}, \bar{r}', t, t') = -\delta(\bar{r} - \bar{r}')\,\delta(t - t')\,, \tag{5}$$

which solution is (in free-space):

$$g(\bar{r}, \bar{r}', t, t') = \begin{cases} \frac{1}{4\pi|\bar{r}-\bar{r}'|}\,\delta(t - t' - \frac{|\bar{r}-\bar{r}'|}{c}) & t > t', \\ 0 & t < t'. \end{cases} \tag{6}$$

From this, the solution to the wave equation for $\bar{A}$ and ϕ can be written as:

$$\bar{A}(\bar{r}, t) = \mu_0 \int_V \mathrm{d}v' \int_{-\infty}^{\infty} \mathrm{d}t'\, \bar{J}(\bar{r}', t')\, g(\bar{r}, \bar{r}', t, t') = \mu_0 \int_V \mathrm{d}v'\, \frac{\bar{J}(\bar{r}', t - R/c)}{4\pi R}\,, \tag{7a}$$

$$\phi(\bar{r}, t) = \int_V \mathrm{d}v'\, \frac{\rho(\bar{r}, t - R/c)}{4\pi\epsilon_0 R}\,, \tag{7b}$$

where $R = |\bar{r} - \bar{r}'|$. These wave equations are known as the time retarded potentials, and essentially say that the potential (either $\bar{A}$ or ϕ) can be calculated at a given point in space $\bar{r}$ and given time t from all previous times.

From these equations, we can calculate the space-time electromagnetic fields:

$$\bar{H}(\bar{r}, t) = \frac{1}{4\pi} \int \mathrm{d}v'\, \nabla \times \frac{\bar{J}(\bar{r}', \tau)}{R}\,, \qquad \tau = t - R/c\,, \tag{8a}$$

$$\bar{E}(\bar{r}, t) = -\frac{1}{4\pi\epsilon_0} \int_V \mathrm{d}v'\, \nabla \frac{\rho(\bar{r}, \tau)}{R} - \frac{\mu_0}{4\pi} \int \mathrm{d}v'\, \frac{\partial}{\partial t} \frac{\bar{J}(\bar{r}', \tau)}{R}\,. \tag{8b}$$

Let us continue with the electric field first:

$$\bar{E}(\bar{r}, t) = -\frac{1}{4\pi\epsilon_0} \int_V \mathrm{d}v' \left[\frac{1}{R}\nabla\rho(\bar{r}', \tau) + \rho(\bar{r}', \tau)\nabla\frac{1}{R}\right] - \frac{\mu_0}{4\pi} \int_V \mathrm{d}v'\, \frac{1}{R}\frac{\partial}{\partial t}\bar{J}(\bar{r}', \tau)\,. \tag{9}$$

At this point, we need to use the following relations:

$$\nabla R = \frac{\bar{R}}{R}\,, \tag{10a}$$

$$\nabla \frac{1}{R} = -\frac{\bar{R}}{R^3}\,, \tag{10b}$$

$$\nabla\rho(\bar{r}', \tau) = \frac{\partial}{\partial\tau}\rho(\bar{r}', \tau)\nabla\tau = -\frac{1}{c}\nabla R \frac{\partial}{\partial\tau}\rho(\bar{r}', \tau) = -\frac{1}{c}\frac{\bar{R}}{R}\frac{\partial}{\partial\tau}\rho(\bar{r}, \tau)\,. \tag{10c}$$

We can therefore continue with the electric field as:

$$\begin{aligned} \bar{E}(\bar{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \mathrm{d}v' \left[\frac{1}{c}\frac{\bar{R}}{R^2}\frac{\partial}{\partial\tau}\rho(\bar{r}', \tau) + \frac{\bar{R}}{R^3}\rho(\bar{r}', \tau)\right] - \frac{\mu_0}{4\pi} \int \mathrm{d}v'\, \frac{1}{R}\frac{\partial}{\partial\tau}\bar{J}(\bar{r}', \tau) \\ &= \frac{1}{4\pi\epsilon_0} \int \mathrm{d}v' \left[\frac{1}{c}\frac{\partial}{\partial\tau}\rho(\bar{r}', \tau) + \frac{1}{R}\rho(\bar{r}', \tau)\right]\frac{\bar{R}}{R^2} - \frac{\mu_0}{4\pi} \int \mathrm{d}v'\, \frac{1}{R}\frac{\partial}{\partial\tau}\bar{J}(\bar{r}', \tau)\,. \end{aligned} \tag{11}$$

We can perform the same type of calculations for the magnetic field using the relation

$$\nabla \times \bar{J}(\bar{r}', \tau) = -\frac{1}{c}\frac{\bar{R}}{R} \times \frac{\partial}{\partial \tau}\bar{J}(\bar{r}', \tau)\,. \tag{12}$$

We get:

$$\bar{H}(\bar{r}, t) = \frac{1}{4\pi}\int \mathrm{d}v' \left[-\frac{1}{c}\frac{\bar{R}}{R^2} \times \frac{\partial}{\partial \tau}\bar{J}(\bar{r}', \tau) - \frac{\bar{R}}{R^3} \times \bar{J}(\bar{r}', \tau)\right]. \tag{13}$$

Upon gathering the expressions for the electric and magnetic field, we eventually get:

$$\bar{E}(\bar{r}, t) = \frac{1}{4\pi}\int \mathrm{d}v' \left\{\left[\frac{1}{c}\frac{\partial}{\partial \tau}\rho(\bar{r}', \tau) + \frac{1}{R}\rho(\bar{r}', \tau)\right]\frac{\bar{R}}{\epsilon_0 R^2} - \frac{\mu_0}{R}\frac{\partial}{\partial \tau}\bar{J}(\bar{r}', \tau)\right\} \tag{14a}$$

$$\bar{H}(\bar{r}, t) = \frac{1}{4\pi}\int \mathrm{d}v' \left[\frac{1}{c}\frac{\partial}{\partial \tau}\bar{J}(\bar{r}', \tau) + \frac{1}{R}\bar{J}(\bar{r}', \tau)\right] \times \frac{\bar{R}}{R^2}\,. \tag{14b}$$

Upon using the boundary conditions for the electric and magnetic field, we construct the integral equations in a standard way:

- EFIE: $\hat{n} \times (\bar{E}^i + \bar{E}^{\mathrm{scat}}) = 0$ on PEC surface

$$\Rightarrow \hat{n} \times \bar{E}^i(\bar{r}, t) + \frac{1}{4\pi}\hat{n} \times \int \mathrm{d}s'\ [\ldots] \tag{15}$$

- MFIE: $\hat{n} \times (\bar{H}^i + \bar{H}^{\mathrm{scat}}) = \bar{J}_s$.

  As we have seen before (in a previous class), this integral equation is expressed in terms of the principal value of the integral with a 1/2 additional factor. Thus:

$$\frac{1}{2}\bar{J}(\bar{r}, t) = \hat{n} \times \bar{H}^i(\bar{r}, t) + \frac{1}{4\pi}\hat{n} \times\ PV\int \mathrm{d}s'\ [\ldots] \tag{16}$$

For the sake of comparison, we can write the MFIE in the frequency domain and in the time domain:

$$\bar{J}(\bar{r}) = 2\hat{n} \times \bar{H}^i(\bar{r}) + 2\hat{n} \times\ PV\int \mathrm{d}s'\ \bar{J}(\bar{r}') \times \nabla' g(\bar{r}, \bar{r}') \qquad \bar{r} \in S\,, \tag{17a}$$

$$\bar{J}(\bar{r}, t) = 2\hat{n} \times \bar{H}^i(\bar{r}) + \frac{1}{2\pi}\hat{n} \times\ PV\int \mathrm{d}s' \left[\frac{1}{c}\frac{\partial}{\partial \tau}\bar{J}(\bar{r}', \tau) + \frac{1}{R}\bar{J}(\bar{r}', \tau)\right] \times \frac{\bar{R}}{R^2}\,. \tag{17b}$$

Note that in the principal value, we essentially exclude the part for which $R = 0$. Since $\tau = t - R/c$ and $R \neq 0$, we always have that $\tau < t$. The time domain equations therefore state that the current at location $\bar{r}$ and time t is equal to a known term $2\hat{n} \times \bar{H}^i(\bar{r}, t)$ plus a term (integral) known from the past history of $\bar{J}$. This is the basis for solving the time domain integral equation by iterative methods, the most well-known one being the *marching-on-in-time*.

3 The marching-on-in-time technique

3.1 General equations

The integral equation can often be cast in the following form:

$$\bar{J}(\bar{r},t) = \bar{J}^i(\bar{r},t) + \int_S \mathrm{d}v' \int_0^{\tau} \mathrm{d}t' \, \overline{\overline{K}}(\bar{r},\bar{r}',t-t') \cdot \bar{J}(\bar{r}',t') \,. \tag{18}$$

eq.10

Note that we have a time integral also as in Eq. (18), $\bar{J}(\bar{r}',t')$ has not yet been set to satisfy any causality condition. Hence, we must then impose $\bar{J}^i(\bar{r},t) = 0$ for $t < 0$, $\bar{r} \in S$.

In order to apply the MoM, we discretize the current both in space and in time:

$$\bar{J}(\bar{r}',t') = \sum_{m'=1}^{M} \sum_{n'=0}^{N} \bar{J}_p(m',n') \; P_s(\bar{r}' - \bar{r}_{m'}) \; P_t(t' - t_{n'}) \,, \tag{19}$$

where P denotes the simple pulse function. In addition, we also apply point-matching, which means that we take the following testing functions:

$$W_{mn}(\bar{r}_m, t_n) = \delta(\bar{r} - \bar{r}_m) \; \delta(t - n\Delta t) = \delta(\bar{r} - \bar{r}_m) \; \delta(t - t_n) \,, \tag{20}$$

where we take $\Delta t = \min\{R_{mm'}/c\}$, $R_{mm'} = |\bar{r}_m - \bar{r}'_m|$. The method is best illustrated on the following example.

3.2 Example

Let us consider a 1D example governed by the following integral equation:

$$g(x,t) = \int_{-x_0}^{x_0} K(x,x') \; f(x',\tau) \mathrm{d}x' \,, \qquad x \in [-x_0, x_0], \quad \tau = \tau(x,x',t) = t - \frac{|x-x'|}{c} \,. \tag{21}$$

Let us chose the following expansion for f:

$$f(x',\tau) \simeq \sum_{i'=1}^{N} \sum_{j'=1}^{J} a_{i'j'} P_{i'j'}(x',\tau) \,, \tag{22}$$

where the pulse basis functions are defined as

$$P_{i'j'} = \begin{cases} 1 & \text{for } x' \in [x_{i'} - \frac{d_x}{2}, x_{i'} + \frac{d_x}{2}] \text{ and } t \in [t_{j'} - \frac{d_t}{2}, t_{j'} + \frac{d_t}{2}] \\ 0 & \text{elsewhere.} \end{cases} \tag{23}$$

Note that we use the definitions: $x_{i'} = i' d_x$, $t_{j'} = j' d_t$, and $d_x = c d_t$. In order to apply point matching, we take the following testing functions:

$$w_{ij}(x,t) = \delta(x - x_i) \, \delta(t - t_j) \,. \tag{24}$$

Upon expanding and testing, we get:

$$\begin{aligned} g(x_i,t_j) = g_{ij} &= \int_{-x_0}^{x_0} K(x_i,x') \sum_{i'=1}^{N} \sum_{j'=1}^{J} a_{i'j'} P_{i'j'}(x',\tau)\delta(t-t_j) \\ &= \sum_{i'=1}^{N} \int_{(i'-\frac{1}{2})d_x}^{(i'+\frac{1}{2})d_x} \mathrm{d}x' \sum_{j'=1}^{J} a_{i'j'} P_{i'j'}(x',\tau) K(id_x,x')\delta(t-t_j)\,. \end{aligned} \tag{25}$$

Coming back to the definition of τ, we write (with the test and the expansion):

$$\tau = t_j - \frac{|x_i - x_{i'}|}{c} = jd_t - |i-i'|\frac{d_x}{c} = (j - |i-i'|)d_t\,, \tag{26}$$

such that the coefficient $a_{i'j'}$ becomes $a_{i',j-|i-i'|}$. The integral equation becomes:

$$g_{ij} = \sum_{i'=1}^{N} a_{i',j-|i-i'|} \int_{(i'-\frac{1}{2})d_x}^{(i'+\frac{1}{2})d_x} \mathrm{d}x' K(id_x,x')\,. \tag{27}$$

We can define the term

$$Z_{ii'} = \int_{(i'-\frac{1}{2})d_x}^{(i'+\frac{1}{2})d_x} \mathrm{d}x' K(id_x,x') \tag{28}$$

and rewrite the previous system as

$$\begin{aligned} g_{ij} &= \sum_{i'=1}^{N} Z_{ii'}\; a_{i',j-|i-i'|} \\ &= Z_{ii}\; a_{ij} + Z_{i,i-1}\; a_{i-1,j-1} + Z_{i,i-2}\; a_{i-2,j-2} + \ldots + Z_{i1}\; a_{1,j-i+1} \\ &\qquad + Z_{i,i+1}\; a_{i+1,j-1} + Z_{i,i+2}\; a_{i+2,j-2} + \ldots + Z_{1,N}\; a_{N,j-N+i}\,. \end{aligned} \tag{29}$$

In this equation, only the first term involves time step j, all the others terms being at $j-1$, $j-2$, ... Therefore, we can solve for a_{ij}:

$$a_{ij} = \frac{1}{Z_{ii}} \left[g_{ij} - \sum_{\substack{i'=1 \\ i' \neq i}}^{N} Z_{ii'}\; a_{i',j-|i-i'|} \right]. \tag{30}$$

The value of all a_{ik} are known for $k < j$, so that a_{ij} is completely specified in closed form by those and the present value of g_{ij}. This process is known as a 1D march-on-in-time approach.

Time-domain MoM is nowadays in its early stage and, although it has been successfully applied to various simple situations, still suffers from numerical instabilities. More work is in progress...