6.642 Continuum Electromechanic
Fall 2008

6.642 — Continuum Electromechanics	Fall 2008
Problem Set 5 - Solutions	
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Problem 7.6.3

Mass conservation requries that

$$\frac{4}{3}\pi\xi_1^3 + \frac{4}{3}\xi_2^3 = 2\left(\frac{4}{3}\pi\xi_0^3\right) \Rightarrow \xi_1^3 + \xi_2^3 = 2\xi_0^3. \tag{1}$$

With the pressure outside the bubbles defined as p_0, the pressure inside the respective bubbles are

$$p_a - p_0 = \frac{2\gamma}{\xi_1}; \quad p_b - p_0 = \frac{2\gamma}{\xi_2} \tag{2}$$

so that the pressure difference driving fluid between the bubbles once the valve is opened is

$$p_a - p_b = 2\gamma\left[\frac{1}{\xi_1} - \frac{1}{\xi_2}\right]. \tag{3}$$

The flow rate between bubbles given by differentiating Eq. 1 is then equal to Q_v and hence to the given expression for the pressure drop through the connecting tubing.

$$Q_v = -\frac{4}{3}\pi 3\xi_1^2\frac{d\xi_1}{dt} = \frac{\pi R^4}{8\eta\ell}(p_a - p_b) = \frac{\pi R^4}{8\eta\ell}2\gamma\left[\frac{1}{\xi_1} - \frac{1}{\xi_2}\right] \tag{4}$$

Thus, the combination of Eqs. 1 and 4 give a first order differential equation describing the evolution of ξ_1 or ξ_2. In normalized terms, that expression is

$$\frac{d\xi_1}{dt} = \frac{1}{\xi_1^2}\left[\frac{1}{(2-\xi_1^3)^{\frac{1}{3}}} - \frac{1}{\xi_1}\right] \tag{5}$$

where

$$\xi_1 = ||\xi_1||\xi_0, \quad t = \underline{t}\left[\frac{16\eta\ell\xi_0^4}{R^4\gamma}\right]$$

Thus, the velocity is a function of ξ_1, and can be pictured as shown in the figure. It is therefore evident that if ξ_1 increases slightly, it will tend to further increase. The static equilibrium at $\xi_1 = \xi_0$ is unstable. Physically this results from the fact that γ is constant. As the radius of curvature of a bubble decreases, the pressure increases and forces the air into the other bubble. Note that this is not what would be found if the bubbles were replaced by most elastic membranes. The example is useful for giving a reminder of what is implied by the concept of a surface tension. Of course, if the bubble can not be modelled as a layer of liquid with interior and exterior interfaces comprised of the same material, then the basic law may not apply. In the figure, note that all variables are normalized. The asymptote comes at the radius where the second bubble has completely collapsed.

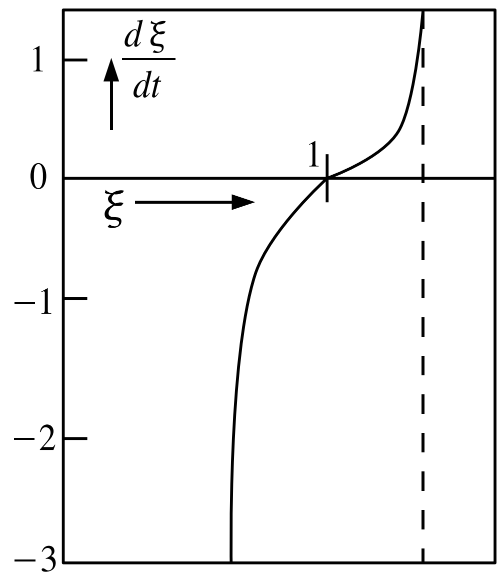

Figure 1: Bubble velocity versus bubble displacement from Eq. (5) (Image by MIT OpenCourseWare.)

Solution to Problem 7.6.3 in *Solutions Manual for Continuum Electromechanics*, 1982, pp. 7.3-7.4

Problem 3.10.3

(a)

The magnetic field is "trapped" in the region between tubes. For an infinitely long pair of coaxial conductors, the field in the annulus is uniform. Hence, because the total flux $\pi a^2 B_0$ must be constant over the length of the system, in the lower region

$$B_z = \frac{a^2 B_0}{a^2 - b^2} \tag{6}$$

(b)

The distribution of surface current is as sketched below. It is determined by the condition that the magnetic flux at the extremities be as found in (a) and by the condition that the normal flux density on any of the perfectly conducting surfaces vanish.

(c)

Using the surface force density $\mathbf{K}\times <\mathbf{B}>$, it is reasonable to expect the net magnetic force in the z direction to be downward.

(d)

One way to find the net force is to enclose the "blob" by the control volume shown in the figure and integrate the stress tensor over the enclosing surface.

$$f_z = \oint_S T_{zj} n_j da$$

Contributions to this integration over surfaces (4) and (2) (the walls of the inner and outer tubes which are perfectly conducting) vanish because there is no shear stress on a perfectly conducting surface. Surface

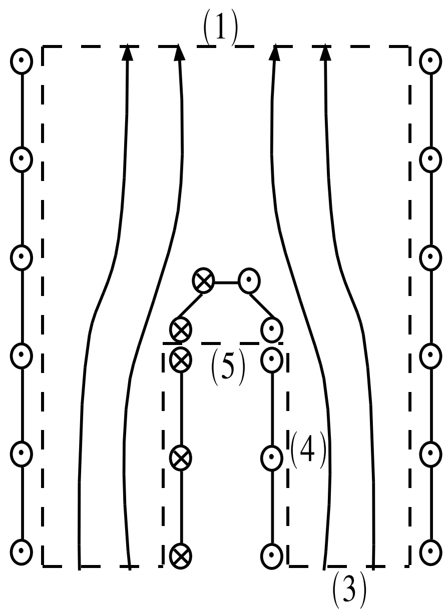

Figure 2: Magnetic field lines and stress tensor enclosing surface (dashed) (Image by MIT OpenCourseWare.)

(5) cuts under the "blob" and hence sustains no magnetic stress. Hence, only surfaces (1) and (3) make contributions, and on them the magnetic flux density is given and uniform. Hence, the net force is

$$f_z = \pi a^2 \left(\frac{B_0^2}{2\mu_0}\right) - \pi(a^2 - b^2)\frac{B_0^2 a^4}{2\mu_0(a^2-b^2)^2} = -\frac{\pi a^2 B_0^2}{2\mu_0}\frac{b^2}{a^2-b^2}. \tag{7}$$

Note that, as expected, this force is negative.

Problem 3.10.4

The electric field is sketched in the figure. The force on the cap should be upward. To find this force use the surface S shown to enclose the cap. On S_1 the field is zero. On S_2 and S_3 the electric shear stress is zero because it is an equi-potential and hence can support no tangential **E**. On S_4 the field is zero. Finally, on S_5 the field is that of infinite coaxial conductors.

$$\mathbf{E} = \mathbf{i_r}\frac{V_0}{\ln\left(\frac{a}{b}\right)}\frac{1}{r}. \tag{8}$$

Thus, the normal electric stress is

$$T_{zz} = -\frac{\epsilon_0}{2}E_r^2 = -\frac{1}{2}\frac{\epsilon_0 V_0^2}{\ln^2\left(\frac{a}{b}\right)}\frac{1}{r^2}, \tag{9}$$

and the integral for the total force reduces to

$$f_z = \oint_S T_{zj}n_j da = -\int_b^a T_{zz}2\pi r dr = \frac{V_0^2\epsilon_0 2\pi}{2\ln^2\frac{a}{b}}\ln\frac{a}{b} = \frac{\pi V_0^2\epsilon_0}{\ln\left(\frac{a}{b}\right)}. \tag{10}$$

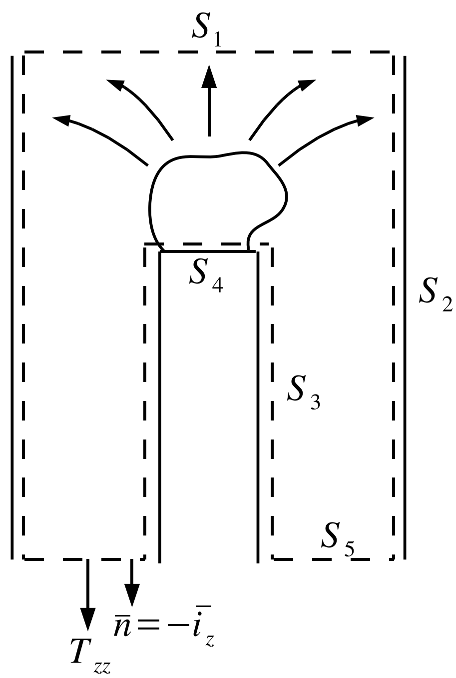

Figure 3: Electric field lines and stress tensor enclosing surface (dashed) (Image by MIT OpenCourseWare.)

Problem 4

(a)

From the results of problem 12.2, we have that the pressure p, acting just to the left of the piston, is

$$p = \frac{\mu_0 I^2}{2w^2}. \tag{11}$$

The exit velocity at each orifice is obtained by using Bernoulli's law just to the left of the piston and at either orifice, from which we obtain

$$V = \left(\frac{\mu_0}{\rho}\right)^{\frac{1}{2}} \frac{I}{w} \tag{12}$$

at each orifice.

(b)

The thrust is

$$T = 2V\frac{dM}{dt} = 2V^2\rho dw. \tag{13}$$

$$T = \frac{2\mu_0 I^2 d}{w}. \tag{14}$$

(c)

In the steady state, we choose to integrate the momentum theorem, Eq. (12.1.29), around a rectangular surface, enclosing the system for $-L \leq x_1 \leq +L$.

$$-\rho V_0^2 a + \rho [V(L)]^2 b = p_0 a - p(L) b + F, \tag{15}$$

where F is the x_1 component force per unit length which the walls exert on the fluid. We see that there is no x_1 component of force from the upper wall, therefore F is the force purely from the lower wall.

In the steady state, conservation of mass, (Eq. 12.1.8), yields

$$V(\ell) = V_0 \frac{a}{b} \mathbf{i_1}. \tag{16}$$

Bernoulli's equation gives us

$$\frac{1}{2}\rho V_0^2 + p_0 = \frac{1}{2}\rho V_0^2 \frac{a^2}{b^2} + P(L). \tag{17}$$

Solving (17) for $P(L)$, and then substituting this result and that of (16) into (15), we finally obtain

$$F = P_0(b-a) + \rho V_0^2 \left(-a + \frac{b}{2} + \frac{a^2}{2b} \right). \tag{18}$$

The problem asked for the force on the lower wall, which is just the negative of F. Thus

$$F_{\text{wall}} = -P_0(b-a) - \rho V_0^2 \left(-a + \frac{a^2}{2b} + \frac{b}{2} \right). \tag{19}$$