6.642 Continuum Electromechanics
Fall 2008

6.642 — Continuum Electromechanics	Fall 2008
Problem Set 7 - Solutions	
Prof. Markus Zahn	MIT OpenCourseWare

Problem 8.12.2

Stress equilibrium at the interface requires that

$$-\Pi - p'_d + p'_e - T_{rr}|_{R+\xi} = 0 \Rightarrow \hat{p}^d = -\mu_0 H_0^2 \frac{\hat{\xi}}{R} + \mu_0 H_0 h_\theta^e; \quad \Pi = \frac{1}{2}\mu_0 H_0^2. \tag{1}$$

Also, at the interface flux is conserved, so

$$\bar{n} \cdot \bar{H}|_{R+\xi} = 0 \Rightarrow \hat{h}_r^e = -j\frac{H_0 m}{R}\hat{\xi}, \tag{2}$$

while at the inner rod surface

$$\hat{h}_r^f = 0. \tag{3}$$

At the outer wall,

$$\hat{\xi}^c = 0. \tag{4}$$

Bulk transfer relations are

$$\begin{bmatrix} \hat{p}^c \\ \hat{p}^d \end{bmatrix} = -\rho\omega^2 \begin{bmatrix} F_m(R,a) & G_m(a,R) \\ G_m(R,a) & F_m(a,R) \end{bmatrix} \begin{bmatrix} 0 \\ \hat{\xi} \end{bmatrix}, \tag{5}$$

$$\begin{bmatrix} \hat{h}_\theta^e \\ \hat{h}_\theta^f \end{bmatrix} = \frac{jm}{R} \begin{bmatrix} F_m(b,R) & G_m(R,b) \\ G_m(b,R) & F_m(R,b) \end{bmatrix} \begin{bmatrix} \hat{h}_r^e \\ 0 \end{bmatrix}. \tag{6}$$

The dispersion equation follows by substituting Eq. (1) for $\hat{p}^d$ in Eq. (5b) with $\hat{h}_\theta^e$ from Eq. (6a). On the right in Eq. (5b), Eq. (2) is substituted. Hence,

$$-\frac{\mu_0 H_0^2}{R}\hat{\xi} + \mu_0 H_0 jm F_m(b,R)\frac{(-jH_0 m)}{R}\hat{\xi} = -\rho\omega^2 F_m(a,R)\hat{\xi}. \tag{7}$$

Thus, the dispersion equation is

$$\omega^2 = \frac{\mu_0 H_0^2}{\rho R F_m(a,R)}\left[1 - \frac{m^2}{R}F_m(b,R)\right]. \tag{8}$$

From the reciprocity energy conditions discussed in Sec. 2.17, $F_m(a,R) > 0$ and $F_m(b,R) < 0$, so Eq. (8) gives real values of ω regardless of k. The system is stable.

Problem 8.13.1

In static equilibrium, the radial stress balance becomes

$$[p] = [T_{rr}] - \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{9}$$

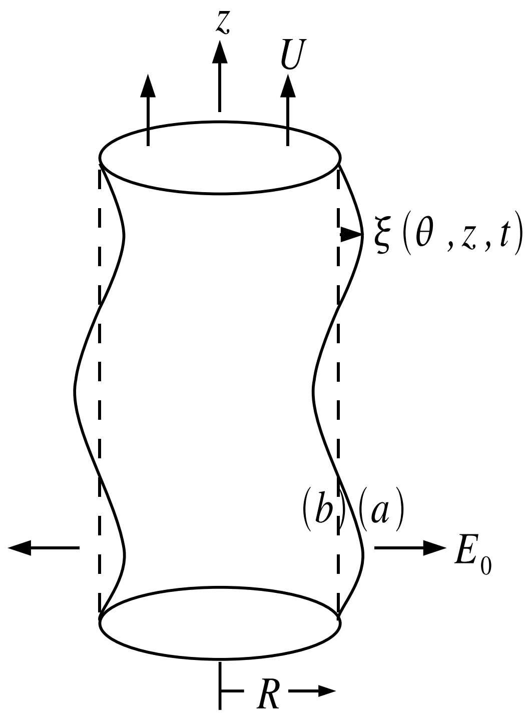

Figure 1: A cylindrical liquid column of equilibrium radius R moves with velocity U in the z direction and has a radial electric field for $r > R + \xi(\theta, z, t)$ (Image by MIT OpenCourseWare.)

so that the pressure jump under this condition is

$$[\Pi] = \frac{1}{2}\epsilon_0 E_0^2 - \frac{\gamma}{R}. \quad (10)$$

In the region surrounding the column, the electric field intensity takes the form

$$\bar{E} = E_0 \frac{R}{r}\bar{i}_r + \bar{e}; \quad \bar{e} = -\nabla\Phi \quad (11)$$

while inside the column the electric field is zero and the pressure is given by

$$p = \Pi_b + p'(r, \theta, z, t) = \Pi_b + \Re[\hat{p}(r)e^{j(\omega t - m\theta - kz)}]. \quad (12)$$

Electrical boundary conditions require that the perturbation potential vanish as r becomes large and that the tangential field vanish on the deformable surface of the column.

$$\bar{n} \times \bar{E}|_{r=R+\xi} \cong \begin{bmatrix} \bar{i}_r & \bar{i}_\theta & \bar{i}_z \\ 1 & -\frac{1}{R}\frac{\partial \xi}{\partial \theta} & -\frac{\partial \xi}{\partial z} \\ E_0\frac{R}{r} + e_r & e_\theta & e_z \end{bmatrix} \Rightarrow e_z = -E_0\frac{\partial \xi}{\partial z}. \quad (13)$$

In terms of complex amplitudes , with $\hat{e}_z = jk\hat{\Phi}$,

$$\hat{\Phi}^a = E_0\hat{\xi}. \quad (14)$$

Stress balance in the radial direction at the interface requires that (with some linearization) ($p'_a \approx 0$)

$$\Pi_a - \Pi_b - p'_b = \frac{1}{2}\epsilon_0 \left[E_0 \frac{R}{R+\xi} + e_r\right]^2 + (T_s)_r. \quad (15)$$

To linear terms, this becomes (Eqs. (f) and (h), Table 7.6.2 for $\bar{T}_s$)

$$\hat{p}_b = \frac{\epsilon_0 E_0^2}{R}\hat{\xi} - \epsilon_0 E_0 \hat{e}_r^a - \frac{\gamma}{R^2}(1 - m^2 - (kR)^2)\hat{\xi}. \tag{16}$$

Bulk relations representing the fields surrouding the column and the fluid within are Eq. (a) of Table 2.16.2 and (f) of Table 7.9.1:

$$\hat{e}_r^a = f_m(\infty, R)\hat{\Phi}^a, \tag{17}$$

$$\hat{p}^b = j(\omega - kU)\rho F_m(0, R)\hat{\vartheta}_r. \tag{18}$$

Recall that $\hat{\vartheta}_r = j(\omega - kU)\hat{\xi}$, and it follows that Eqs. (17), (18) and (14) can be substituted into the stress balance equation to obtain

$$-(\omega - kU)^2 \rho F_m(0, R)\hat{\xi} = \frac{\epsilon_0 E_0^2}{R}\hat{\xi} - \epsilon_0 E_0^2 f_m(\infty, R)\hat{\xi} - \frac{\gamma}{R^2}(1 - m^2 - k^2 R^2)\hat{\xi}. \tag{19}$$

If the amplitude is to be finite, the coefficients must equilibrate. The result is the dispersion equation given with the problem.

Solution to Problem 8.13.1 in Melcher, James R. *Solutions Manual for Continuum Electromechanics*, 1982, pp. 8.23-8.24.