6.642 Continuum Electromechanics
Fall 2008

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Kelvin-Helmholtz Instability

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Problem Statement

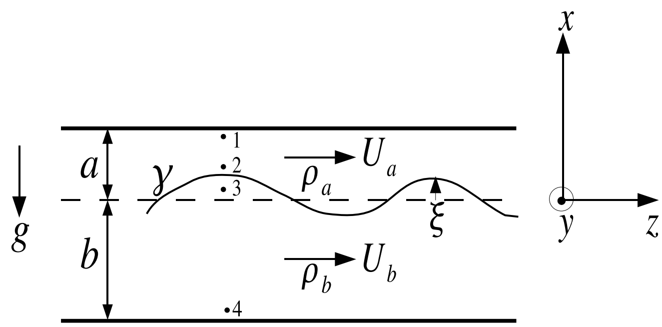

Figure 1: Geometry for Kelvin-Helmholtz instability problem of two superposed fluids moving at different z-directed velocities (Image by MIT OpenCourseWare.)

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = \frac{\rho j(\omega - k_z U)}{k} \begin{bmatrix} -\coth k\Delta & \frac{1}{\sinh k\Delta} \\ -\frac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{v}_x^{\alpha} \\ \hat{v}_x^{\beta} \end{bmatrix}$$

$$k = (k_y^2 + k_z^2)^{1/2}$$

Equilibrium

$$\mathbf{v} = U\mathbf{i_z}, \quad \xi = 0$$

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = -\nabla p_{eq} - \rho g \mathbf{i_x}$$

$$-\frac{dp_{eq}}{dx} = \rho g \Rightarrow p_{eq} = -\rho g x + \text{constant}$$

$$p_{eq}(x) = \begin{cases} -\rho_a g x + p_0 & x > 0 \\ -\rho_b g x + p_0 & x < 0 \end{cases}$$

Perturbations

$$v_x = \frac{\partial \xi}{\partial t} + v_y \frac{\partial \xi}{\partial y} + v_z \frac{\partial \xi}{\partial z} \Rightarrow v_x = \frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial z}$$

$$\xi(x, z, t) = \Re \left[\hat{\xi} e^{j(\omega t - k_y y - k_z z)} \right] \Rightarrow \hat{v}_x = j(\omega - k_z U)\hat{\xi}$$

$$\hat{v}_{x2} = j(\omega - k_z U_a)\hat{\xi}; \quad \hat{v}_{x3} = j(\omega - k_z U_b)\hat{\xi}$$

Boundary Conditions: $\hat{v}_{x1} = \hat{v}_{x4} = 0$

$$\hat{p}_2' = \frac{\rho_a j}{k}(\omega - k_z U_a)\coth ka \hat{v}_{x2} = -\frac{\rho_a}{k}(\omega - k_z U_a)^2 \coth ka \hat{\xi}$$

$$\hat{p}_3' = \frac{\rho_b j}{k}(\omega - k_z U_b)\coth kb \hat{v}_{x3} = \frac{\rho_b}{k}(\omega - k_z U_b)^2 \coth kb \hat{\xi}$$

$$p_2(x = 0 + \xi) = p_{2eq}(0 + \xi) + p_2' = \underbrace{p_{2eq}(0)}_{p_0} + \underbrace{\left.\frac{dp_{2eq}}{dx}\right|_{x=0} \xi + p_2'}_{p_2}$$

$$\hat{p}_2 = \hat{p}_2' - \rho_a g \hat{\xi} = -\left[\frac{\rho_a}{k}(\omega - k_z U_a)^2 \coth ka + \rho_a g\right]\hat{\xi}$$

$$p_3(x = 0 + \xi) = p_{3eq}(0 + \xi) + p_3' = \underbrace{p_{3eq}(0)}_{p_0} + \underbrace{\left.\frac{dp_{3eq}}{dx}\right|_{x=0} \xi + p_3'}_{p_3}$$

$$\hat{p}_3 = \hat{p}_3' - \rho_b g \hat{\xi} = \left[\frac{\rho_b}{k}(\omega - k_z U_b)^2 \coth kb - \rho_b g\right]\hat{\xi}$$

Interfacial Force Balance

$$\hat{p}_3 - \hat{p}_2 - \gamma(k^2)\hat{\xi} = 0$$

$$\left[\frac{\rho_b}{k}(\omega - k_z U_b)^2 \coth kb - \rho_b g\right]\hat{\xi} + \left[\frac{\rho_a}{k}(\omega - k_z U_a)^2 \coth ka + \rho_a g\right]\hat{\xi} - \gamma k^2 \hat{\xi} = 0$$

Dispersion Relation

$$\frac{\rho_a}{k}(\omega - k_z U_a)^2 \coth ka + \frac{\rho_b}{k}(\omega - k_z U_b)^2 \coth kb - g(\rho_b - \rho_a) - \gamma k^2 = 0$$

$$\frac{\rho_a}{k}\coth ka\left[\omega^2 - 2k_z U_a \omega + (k_z U_a)^2\right] + \frac{\rho_b}{k}\coth kb\left[\omega^2 - 2k_z U_b \omega + (k_z U_b)^2\right] - g(\rho_b - \rho_a) - \gamma k^2 = 0$$

$$\frac{\omega^2}{k}(\rho_a \coth ka + \rho_b \coth kb) - \frac{2\omega k_z}{k}(\rho_a U_a \coth ka + \rho_b U_b \coth kb)$$
$$+ \frac{k_z^2}{k}\left(\rho_a \coth ka U_a^2 + \rho_b \coth kb U_b^2\right) - g(\rho_b - \rho_a) - \gamma k^2 = 0$$

$$A = \frac{1}{k}(\rho_a \coth ka + \rho_b \coth kb)$$

$$B = \frac{2k_z}{k}(\rho_a U_a \coth ka + \rho_b U_b \coth kb)$$

$$C = \frac{k_z^2}{k}\left(\rho_a \coth ka U_a^2 + \rho_b \coth kb U_b^2\right) - g(\rho_b - \rho_a) - \gamma k^2$$

$$A\omega^2 - B\omega + C = 0$$

$$\omega = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

Unstable if $B^2 - 4AC < 0$

$$\begin{aligned}
B^2 - 4AC =& 4\left(\frac{k_z}{k}\right)^2 (\rho_a U_a \coth ka + \rho_b U_b \coth kb)^2 \\
&- 4\left(\frac{k_z}{k}\right)^2 (\rho_a \coth ka + \rho_b \coth kb)(\rho_a U_a^2 \coth ka + \rho_b U_b^2 \coth kb) \\
&+ \frac{4}{k}[g(\rho_b - \rho_a) + \gamma k^2](\rho_a \coth ka + \rho_b \coth kb) \\
=& 4\left(\frac{k_z}{k}\right)^2 [\rho_a^2 U_a^2 \coth^2 ka + \rho_b^2 U_b^2 \coth^2 kb + 2\rho_a\rho_b U_a U_b \coth ka \coth kb \\
&- \rho_a^2 U_a^2 \coth^2 ka - \rho_b^2 U_b^2 \coth^2 kb - \rho_a\rho_b \coth ka \coth kb (U_a^2 + U_b^2)] \\
&+ \frac{4}{k}[g(\rho_b - \rho_a) + \gamma k^2][\rho_a \coth ka + \rho_b \coth kb] \\
=& 4\left(\frac{k_z}{k}\right)^2 [\rho_a\rho_b \coth ka \coth kb (2U_a U_b - U_a^2 - U_b^2)] \\
&+ \frac{4}{k}[g(\rho_b - \rho_a) + \gamma k^2][\rho_a \coth ka + \rho_b \coth kb] \\
=& \frac{4}{k}\left[-\frac{k_z^2}{k}\rho_a\rho_b \coth ka \coth kb (U_a - U_b)^2 + (g(\rho_b - \rho_a) + \gamma k^2)(\rho_a \coth ka + \rho_b \coth kb)\right]
\end{aligned}$$

Unstable if $B^2 - 4AC < 0$

$$-\frac{k_z^2}{k^2}\rho_a\rho_b \coth ka \coth kb (U_a - U_b)^2 < -[g(\rho_b - \rho_a) + \gamma k^2](\rho_a \coth ka + \rho_b \coth kb)$$

$$(U_a - U_b)^2 > [g(\rho_b - \rho_a) + \gamma k^2]\frac{\rho_a \coth ka + \rho_b \coth kb}{\rho_a\rho_b \coth ka \coth kb}\frac{k^2}{k_z^2}(\text{Unstable})$$

Special Case: $ka \gg 1, \quad kb \gg 1, \quad (\coth ka = \coth kb = 1)$

$$(U_a - U_b)^2 > [g(\rho_b - \rho_a) + \gamma k^2]\left(\frac{\rho_a + \rho_b}{\rho_a\rho_b}\right)\frac{k^2}{k_z^2}(\text{Unstable})$$