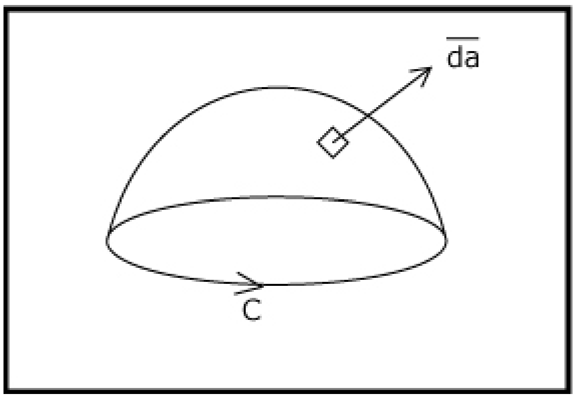
6.642 Continuum Electromechanics
Fall 2008

6.642, Continuum Electromechanics
Prof. Markus Zahn
Lecture 1: Review of Maxwell's Equations

I. Maxwell's Equations in Integral Form in Free Space

1. Faraday's Law

$$\underbrace{\oint_C \bar{E} \bullet \overline{ds}}_{\text{Circulation of } \bar{E}} = -\frac{d}{dt} \underbrace{\int_S \mu_0 \bar{H} \bullet \overline{da}}_{\text{Magnetic Flux}}$$



$\mu_0 = 4\pi \times 10^{-7}$ henries/m [magnetic permeability of free space]

EQS form: $\oint_C \bar{E} \bullet \overline{ds} = 0$ (Kirchoff's Voltage Law, conservative electric field)

MQS circuit form: $v = L\frac{di}{dt}$ (Inductor)

2. Ampère's Law (with Displacement Current)

$$\underbrace{\oint_C \bar{H} \bullet \overline{ds}}_{\text{Circulation of } \bar{H}} = \underbrace{\int_S \bar{J} \bullet \overline{da}}_{\text{Conduction Current}} + \underbrace{\frac{d}{dt} \int_S \varepsilon_0 \bar{E} \bullet \overline{da}}_{\text{Displacement Current}}$$

MQS form: $\oint_C \bar{H} \bullet \overline{ds} = \int_S \bar{J} \bullet \overline{da}$

EQS circuit form: $i = C\frac{dv}{dt}$ (capacitor)

3. Gauss' Law for Electric Field

$$\oint_S \varepsilon_0 \bar{E} \cdot \overline{da} = \int_V \rho \, dV$$

$$\varepsilon_0 \approx \frac{10^{-9}}{36\pi} \approx 8.854 \times 10^{-12} \text{ farads/m}$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \approx 3 \times 10^8 \text{m/s} \text{ (Speed of electromagnetic waves in free space)}$$

4. Gauss' Law for Magnetic Field

$$\oint_S \mu_0 \bar{H} \cdot \overline{da} = 0$$

In free space:

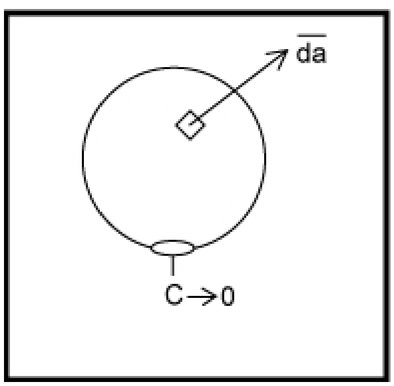
$$\bar{B} = \mu_0 \bar{H}$$

$\bar{B}$: magnetic flux density

$\bar{H}$: magnetic field intensity

5. Conservation of Charge

Take Ampère's Law with displacement current and let contour $C \rightarrow 0$



$$\lim_{C \to 0} \oint_C \bar{H} \cdot \overline{ds} = 0 = \oint_S \bar{J} \cdot \overline{da} + \frac{d}{dt} \underbrace{\oint_S \varepsilon_0 \bar{E} \cdot \overline{da}}_{\int_V \rho \, dV}$$

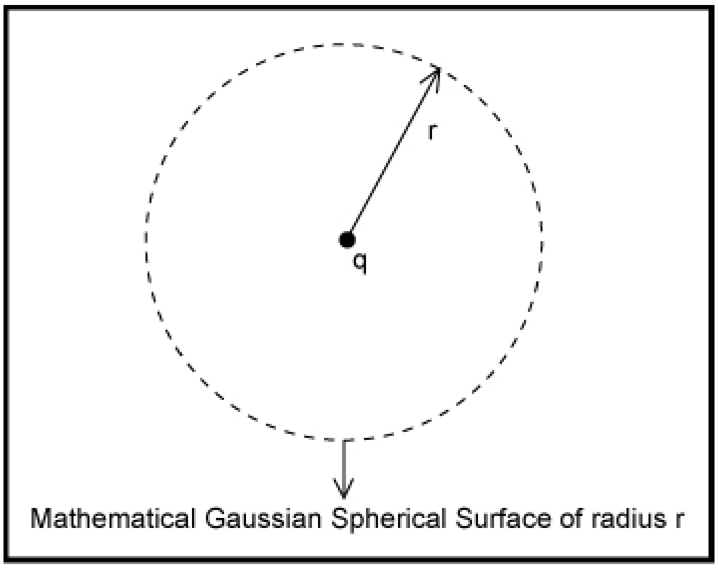
$$\underbrace{\oint_S \bar{J} \cdot \overline{da}}_{\text{Total current leaving volume}} + \frac{d}{dt} \underbrace{\int_V \rho \, dV}_{\text{Total charge inside volume}} = 0$$

through surface

6. Lorentz Force Law – Force on moving point charge in free space at velocity $\overline{v}$ in electric field $\overline{E}$ and magnetic field $\overline{H}$.

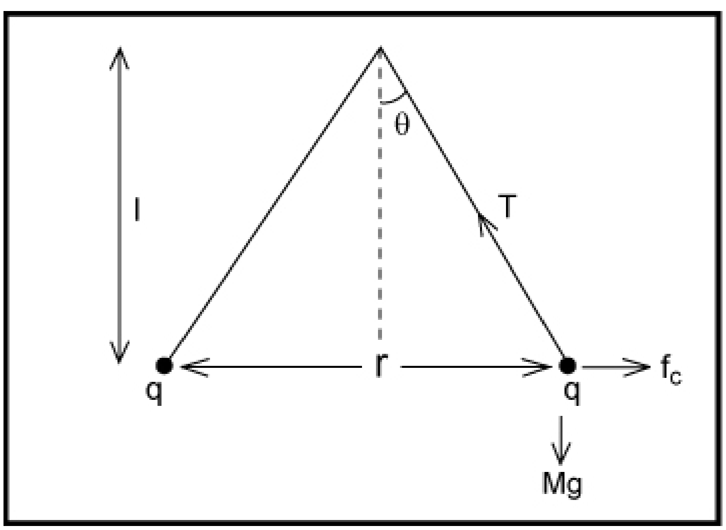
$$\overline{f} = q\left(\overline{E} + \overline{v} \times \mu_0 \overline{H}\right)$$

II. <u>Electric Field from Point Charge</u>



$$\oint_S \varepsilon_0 \overline{E} \cdot \overline{da} = \varepsilon_0 E_r 4\pi r^2 = q$$

$$E_r = \frac{q}{4\pi \varepsilon_0 r^2}$$



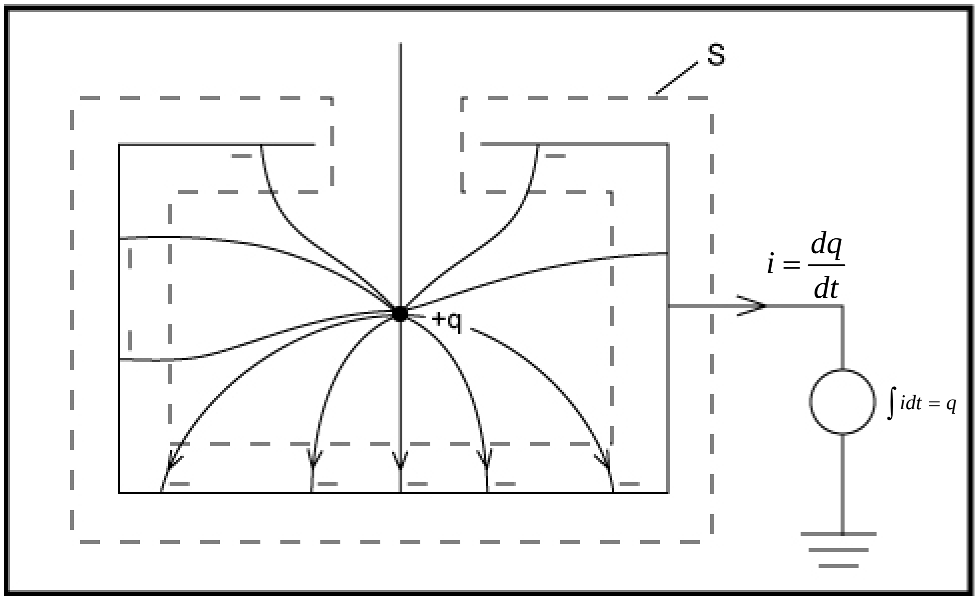
$$T \sin\theta = f_c = \frac{q^2}{4\pi \varepsilon_0 r^2}$$

$$T \cos\theta = Mg$$

$$\tan\theta = \frac{q^2}{4\pi \varepsilon_0 r^2 Mg} = \frac{r}{2l}$$

$$q = \left[\frac{2\pi \varepsilon_0 r^3 Mg}{l}\right]^{1/2}$$

III. Faraday Cage



$$\oint_S \bar{J} \cdot \overline{da} = i = -\frac{d}{dt}\int \rho\, dV = -\frac{d}{dt}(-q) = \frac{dq}{dt}$$

$$\int idt = q$$

IV. Divergence Theorem

1. Divergence Operation

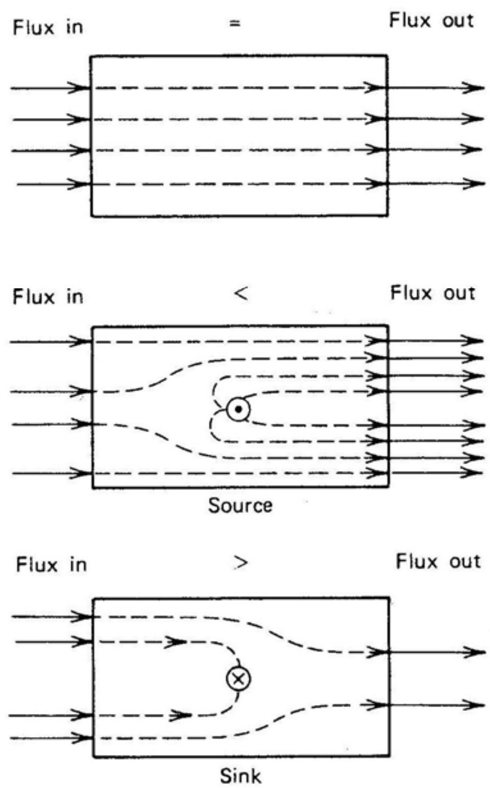


Figure 1-13 The net flux through a closed surface tells us whether there is a source or sink within an enclosed volume.

$$\oint_S \overline{A} \cdot \overline{dS} = \int_V \text{div}\left(\overline{A}\right) dV$$

$$\text{div}\,\overline{A} = \lim_{\Delta V \to 0} \frac{\oint_S \overline{A} \cdot \overline{dS}}{\Delta V}$$

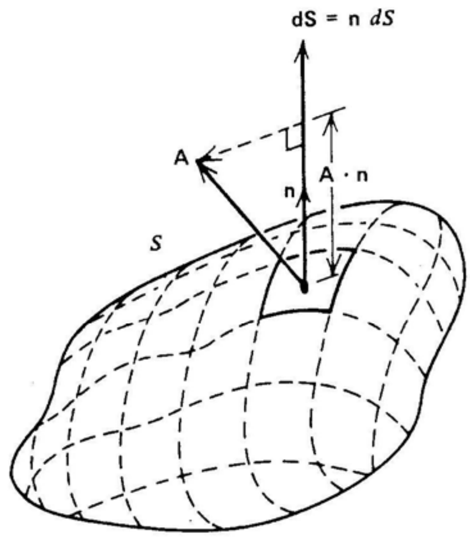


Figure 1-14 The flux of a vector **A** through the closed surface S is given by the surface integral of the component of **A** perpendicular to the surface S. The differential vector surface area element **dS** is in the direction of the unit normal **n**.

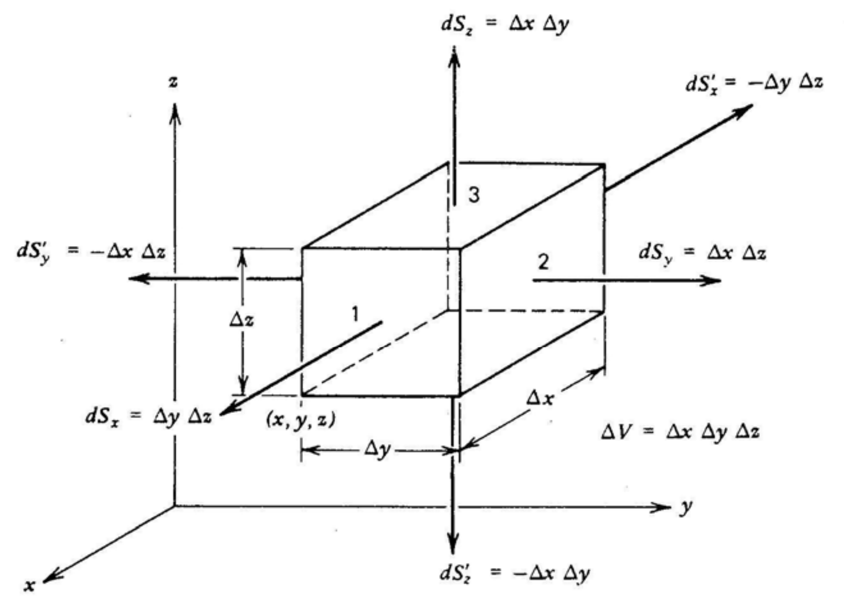


Figure 1-15 Infinitesimal rectangular volume used to define the divergence of a vector.

$$\Phi = \int \overline{A} \bullet \overline{ds} = \int_1 A_x (x,y,z)\,dydz - \int_{1'} A_x (x-\Delta x,y,z)\,dydz$$
$$+ \int_2 A_y (x,y+\Delta y,z)\,dxdz - \int_{2'} A_y (x,y,z)\,dxdz$$
$$+ \int_3 A_z (x,y,z+\Delta z)\,dxdy - \int_{3'} A_z (x,y,z)\,dxdy$$

$$\Phi \approx \Delta x \Delta y \Delta z \left\{ \frac{\left[A_x (x,y,z) - A_x (x-\Delta x,y,z)\right]}{\Delta x} + \frac{\left[A_y (x,y+\Delta y,z) - A_y (x,y,z)\right]}{\Delta y} + \frac{\left[A_z (x,y,z+\Delta z) - A_z (x,y,z)\right]}{\Delta z} \right\}$$

$$\approx \Delta V \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right]$$

$$\text{div } \overline{A} = \lim_{\Delta V \to 0} \frac{\oint_S \overline{A} \bullet \overline{dS}}{\Delta V} = \frac{\Phi}{\Delta V} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Del Operator: $\nabla = \overline{i}_x \frac{\partial}{\partial x} + \overline{i}_y \frac{\partial}{\partial y} + \overline{i}_z \frac{\partial}{\partial z}$

$$\text{div } \overline{A} = \nabla \bullet \overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

2. Gauss' Integral Theorem

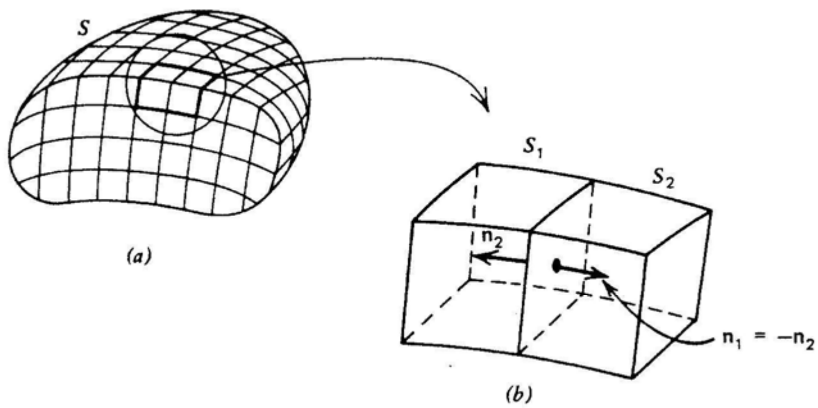


Figure 1-17 Nonzero contributions to the flux of a vector are only obtained across those surfaces that bound the outside of a volume. (a) Within the volume the flux leaving one incremental volume just enters the adjacent volume where (b) the outgoing normals to the common surface separating the volumes are in opposite directions.

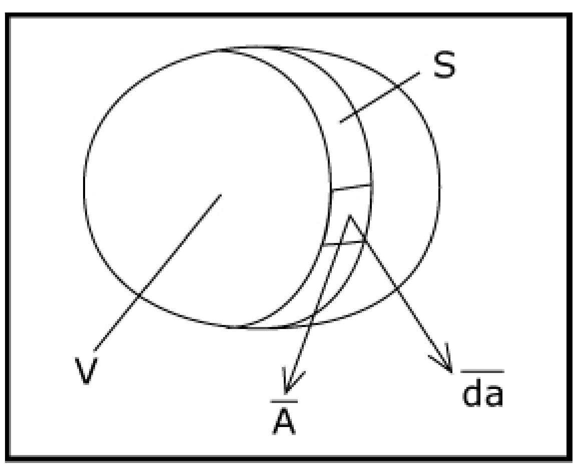
Courtesy of Krieger Publishing. Used with permission.

$$\oint_S \overline{A} \cdot \overline{dS} = \sum_{\substack{i=1 \\ N\to\infty}}^{N} \oint_{dS_i} \overline{A} \cdot \overline{dS_i}$$

$$= \lim_{\substack{N\to\infty \\ \Delta V_n \to 0}} \sum_{i=1}^{N} \left(\nabla \cdot \overline{A}\right) \Delta V_i$$

$$= \int_V \nabla \cdot \overline{A}\, dV$$

$$\int_V \nabla \cdot \overline{A}\, dV = \oint_S \overline{A} \cdot \overline{da}$$



3. Gauss' Law in Differential Form

$$\oint_S \varepsilon_0\, \overline{E} \cdot \overline{da} = \int_V \nabla \cdot \left(\varepsilon_0 \overline{E}\right) dV = \int_V \rho\, dV$$

$$\nabla \cdot \left(\varepsilon_0 \overline{E}\right) = \rho$$

$$\oint_S \mu_0\ \overline{H} \cdot \overline{da} = \int_V \nabla \cdot \left(\mu_0 \overline{H}\right) dV = 0$$

$$\nabla \cdot \left(\mu_0 \overline{H}\right) = 0$$

V. <u>Stokes' Theorem</u>

1. Curl Operation

$$\oint_C \overline{A} \cdot \overline{ds} = \int_S \text{Curl}\left(\overline{A}\right) \cdot \overline{da}$$

$$\text{Curl}\left(\overline{A}\right)_n = \lim_{da_n \to 0} \frac{\oint_C \overline{A} \cdot \overline{ds}}{da_n}$$

STOKES' THEOREM

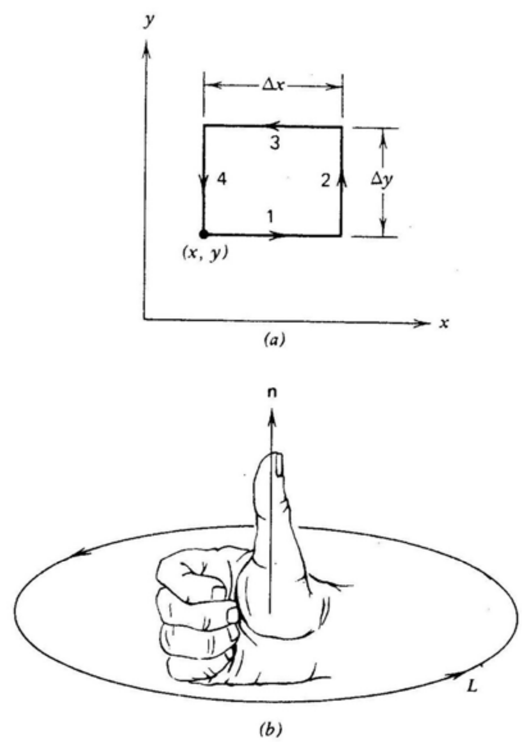


Figure 1-19 (a) Infinitesimal rectangular contour used to define the circulation. (b) The right-hand rule determines the positive direction perpendicular to a contour.

Courtesy of Krieger Publishing. Used with permission.

$$\oint_C \overline{A} \cdot \overline{ds} = \int_{\underset{1}{x}}^{x+\Delta x} A_x(x,y)\,dx + \int_{\underset{2}{y}}^{y+\Delta y} A_y(x+\Delta x, y)\,dy + \int_{\underset{3}{x+\Delta x}}^{x} A_x(x, y+\Delta y)\,dx$$

$$+ \int_{\underset{4}{y+\Delta y}}^{y} A_y(x,y)\,dy$$

$$= \Delta x \Delta y \left[\frac{\left[A_x(x,y) - A_x(x, y+\Delta y)\right]}{\Delta y} + \frac{\left[A_y(x+\Delta x, y) - A_y(x,y)\right]}{\Delta x} \right]$$

$$= da_z \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]$$

$$\text{Curl}\left(\overline{A}\right)_z = \frac{\oint \overline{A} \cdot \overline{ds}}{da_z} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

By symmetry

$$\text{Curl}\left(\overline{A}\right)_y = \frac{\oint \overline{A} \cdot \overline{ds}}{da_y} = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$\text{Curl}\left(\overline{A}\right)_x = \frac{\oint \overline{A} \cdot \overline{ds}}{da_x} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$\text{Curl}\,\overline{A} = \overline{i}_x \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] + \overline{i}_y \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] + \overline{i}_z \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]$$

$$= \det \begin{bmatrix} \overline{i}_x & \overline{i}_y & \overline{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

$$= \nabla \times \overline{A}$$

2. Stokes' Integral Theorem

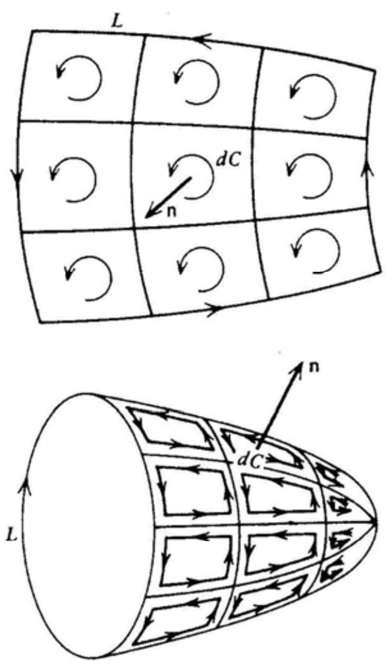


Figure 1-23 Many incremental line contours distributed over any surface, have nonzero contribution to the circulation only along those parts of the surface on the boundary contour L.

Courtesy of Krieger Publishing. Used with permission.

$$\lim_{N\to\infty}\sum_{i=1}^{N}\oint_{dC_i}\overline{A}\cdot\overline{ds_i} = \oint_{C}\overline{A}\cdot\overline{ds}$$

$$= \sum_{i=1}^{N\to\infty}\left(\nabla\times\overline{A}\right)\cdot\overline{da_i}$$

$$= \int_{S}\left(\nabla\times\overline{A}\right)\cdot\overline{da}$$

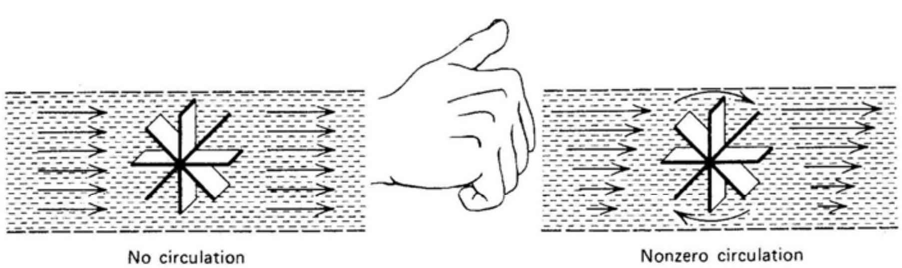


Figure 1-20 A fluid with a velocity field that has a curl tends to turn the paddle wheel. The curl component found is in the same direction as the thumb when the fingers of the right hand are curled in the direction of rotation.

Courtesy of Krieger Publishing. Used with permission.

3. Faraday's Law in Differential Form

$$\oint_C \overline{E} \cdot \overline{ds} = \int_S \left(\nabla \times \overline{E}\right) \cdot \overline{da} = -\frac{d}{dt} \int_S \mu_0 \overline{H} \cdot \overline{da}$$

$$\nabla \times \overline{E} = -\mu_0 \frac{\partial \overline{H}}{\partial t}$$

4. Ampère's Law in Differential Form

$$\oint_C \overline{H} \cdot \overline{ds} = \int_S \nabla \times \overline{H} \cdot \overline{da} = \int_S \overline{J} \cdot \overline{da} + \frac{d}{dt} \int_S \varepsilon_0 \overline{E} \cdot \overline{da}$$

$$\nabla \times \overline{H} = \overline{J} + \varepsilon_0 \frac{\partial \overline{E}}{\partial t}$$

VI. Applications to Maxwell's Equations

1. Vector Identity

$$\lim_{C \to 0} \oint_C \overline{A} \cdot \overline{ds} = 0 = \oint_S \left(\nabla \times \overline{A}\right) \cdot \overline{da} = \int_V \nabla \cdot \left(\nabla \times \overline{A}\right) dV$$

$$\nabla \cdot \left(\nabla \times \overline{A}\right) = 0$$

2. Charge Conservation

$$\nabla \cdot \left\{\nabla \times \overline{H} = \overline{J} + \varepsilon_0 \frac{\partial \overline{E}}{\partial t}\right\}$$

$$0 = \nabla \cdot \left[\overline{J} + \varepsilon_0 \frac{\partial \overline{E}}{\partial t}\right]$$

$$0 = \nabla \cdot \overline{J} + \frac{\partial \rho}{\partial t}$$

3. Magnetic Field

$$\nabla \cdot \left\{\nabla \times \overline{E} = -\mu_0 \frac{\partial \overline{H}}{\partial t}\right\}$$

$$0 = -\frac{\partial}{\partial t}\left[\nabla \cdot \mu_0 \overline{H}\right] \Rightarrow \nabla \cdot \left(\mu_0 \overline{H}\right) = 0$$

VII. Boundary Conditions

1. Gauss' Continuity Condition

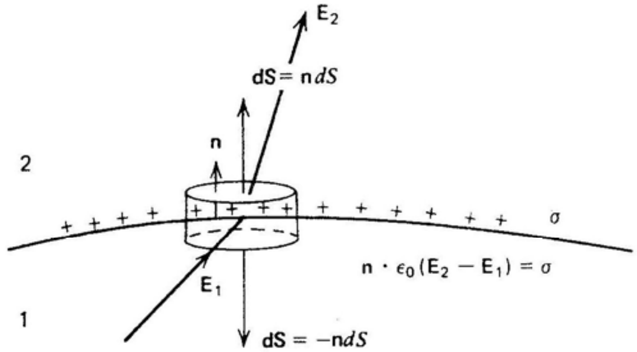


Figure 2-19 Gauss's law applied to a differential sized pill-box surface enclosing some surface charge shows that the normal component of $\varepsilon_0 \mathbf{E}$ is discontinuous in the surface charge density.

Courtesy of Krieger Publishing. Used with permission.

$$\oint_S \varepsilon_0 \bar{E} \cdot \overline{da} = \int_S \sigma_s dS \Rightarrow \varepsilon_0 \left(E_{2n} - E_{1n}\right) dS = \sigma_s dS$$

$$\varepsilon_0 \left(E_{2n} - E_{1n}\right) = \sigma_s \Rightarrow \bar{n} \cdot \left[\varepsilon_0 \left(\bar{E}_2 - \bar{E}_1\right)\right] = \sigma_s$$

2. Continuity of Tangential $\bar{E}$

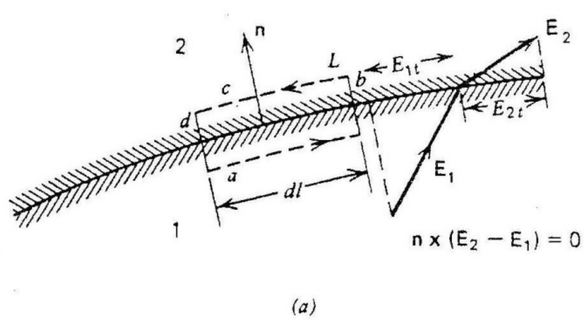


Figure 3-12 (*a*) Stokes' law applied to a line integral about an interface of discontinuity shows that the tangential component of electric field is continuous across the boundary.

Courtesy of Krieger Publishing. Used with permission.

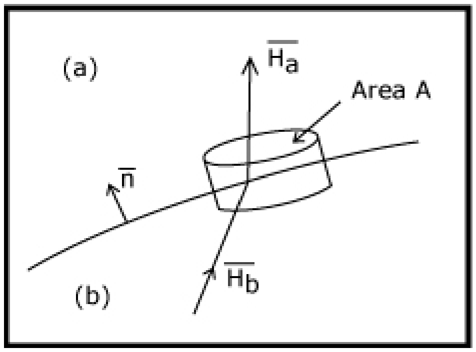
$$\oint_C \bar{E} \cdot \overline{ds} = \left(E_{1t} - E_{2t}\right) dl = 0 \Rightarrow E_{1t} - E_{2t} = 0$$

$$\bar{n} \times \left(\bar{E}_1 - \bar{E}_2\right) = 0$$

Equivalent to $\Phi_1 = \Phi_2$ along boundary (Electric potential is continuous at a boundary)

3. Normal H

$$\nabla \cdot \mu_0 \overline{H} = 0 \Rightarrow \oint_S \mu_0 \overline{H} \cdot \overline{da} = 0$$



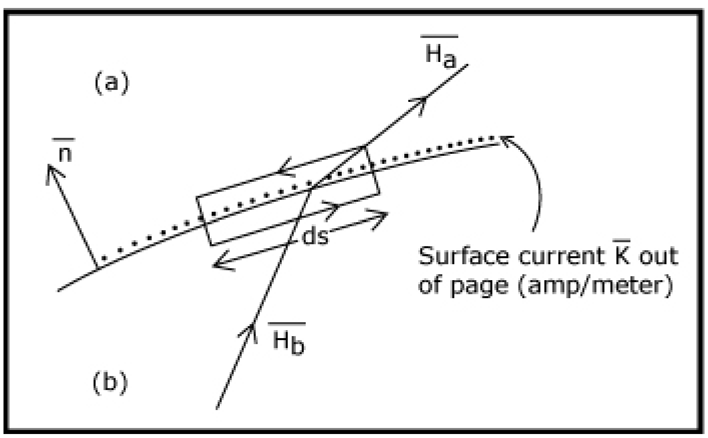
$$\mu_0 \left(H_{an} - H_{bn}\right) A = 0$$

$$H_{an} = H_{bn}$$

$$\overline{n} \cdot \left[\overline{H}_a - \overline{H}_b\right] = 0$$

4. Tangential $\overline{H}$

$$\nabla \times \overline{H} = \overline{J} \Rightarrow \oint_C \overline{H} \cdot \overline{ds} = \int_S \overline{J} \cdot \overline{da}$$

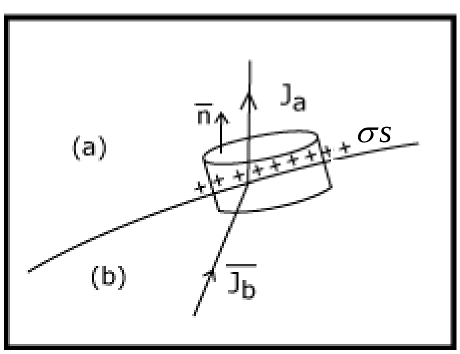


$$H_{bt} \cancel{ds} - H_{at} \cancel{ds} = K \cancel{ds}$$

$$H_{bt} - H_{at} = K$$

$$\overline{n} \times \left[\overline{H}_a - \overline{H}_b\right] = \overline{K}$$

5. Conservation of Charge Boundary Condition



$$\nabla \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\oint_S \overline{J} \cdot \overline{da} + \frac{d}{dt} \int_V \rho dV = 0$$

$$\overline{n} \cdot \left[\overline{J}_a - \overline{J}_b\right] + \frac{\partial \sigma_s}{\partial t} = 0$$

Equivalently: $\overline{n} \cdot [\overline{J}_a + \varepsilon_0 \frac{\partial E_a}{\partial t} - (\overline{J}_b + \varepsilon_0 \frac{\partial E_a}{\partial t})] = 0$

VIII. Poisson's and Laplace's Equations

1. Poisson's Equation

$$\nabla \times \bar{E} = 0 \Rightarrow \bar{E} = -\nabla\Phi$$

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla \cdot (\nabla\Phi) = \nabla^2\Phi = -\frac{\rho}{\varepsilon_0} \quad \text{(Poisson's Equation)}$$

2. Particular and Homogeneous Solutions

$$\nabla^2\Phi_P = -\frac{\rho}{\varepsilon_0} \qquad \text{Poisson's Equation} \Rightarrow \Phi_P(\bar{r}) = \int_{V'} \frac{\rho(\bar{r}')\,dV'}{4\pi\varepsilon_0 |\bar{r} - \bar{r}'|}$$

$$\nabla^2\Phi_h = 0 \qquad \text{Laplace's Equation}$$

$$\nabla^2(\Phi_P + \Phi_h) = -\frac{\rho}{\varepsilon_0}$$

$\Phi = \Phi_P + \Phi_h$ must satisfy boundary conditions

3. Uniqueness of Solutions

Try 2 solutions Φ_a and Φ_b

$$\nabla^2\Phi_a = -\frac{\rho}{\varepsilon_0}$$

$$\nabla^2\Phi_b = -\frac{\rho}{\varepsilon_0}$$

$$\nabla^2(\Phi_a - \Phi_b) = 0$$

$$\Phi_d = \Phi_a - \Phi_b \Rightarrow \nabla^2\Phi_d = 0 \Rightarrow \Phi_d = 0$$

$$\nabla \cdot [\Phi_d \nabla\Phi_d] = \Phi_d \underbrace{\nabla^2\Phi_d}_{0} + \nabla\Phi_d \cdot \nabla\Phi_d = |\nabla\Phi_d|^2$$

$$\int_V \nabla \cdot [\Phi_d \nabla\Phi_d]\,dV = \oint_S \Phi_d \nabla\Phi_d \cdot \overline{da} = \int_V |\nabla\Phi_d|^2\,dV = 0$$

On S, $\Phi_d = 0$ or $\nabla\Phi_d \cdot \overline{da} = 0$

$\Phi_d = 0 \Rightarrow \Phi_a = \Phi_b$ on S

$\nabla\Phi_d \cdot \overline{da} = 0 \Rightarrow \frac{\partial\Phi_a}{\partial n} = \frac{\partial\Phi_b}{\partial n}$ on S $\Rightarrow E_{na} = E_{nb}$ on S

A problem is uniquely posed when the potential or the normal derivative of the potential (normal component of electric field) is specified on the surface surrounding the volume.

IX. Two-Dimensional Solutions to Laplace's Equation in Cartesian Coordinates, $\Phi(x, y)$

$$\nabla^2 \Phi(x, y) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

1. Try product solution: $\Phi(x, y) = X(x)\, Y(y)$

$$Y(y) \frac{d^2 X(x)}{dx^2} + X(x) \frac{d^2 Y(y)}{dy^2} = 0$$

Multiply through by $\frac{1}{XY}$:

$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{\text{only a function of x}} = \underbrace{-\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{\text{only a function of y}} = \underbrace{-k^2}$$

k=separation constant

$$\frac{d^2 X}{dx^2} = -k^2 X \quad ; \quad \frac{d^2 Y}{dy^2} = k^2 Y$$

2. Zero Separation Constant Solutions: k=0

$$\frac{d^2 X}{dx^2} = 0 \Rightarrow X = a_1 x + b_1$$

$$\frac{d^2 Y}{dy^2} = 0 \Rightarrow Y = c_1 y + d_1$$

$$\Phi(x, y) = XY = a_2 + b_2 x + c_2 y + d_2 xy$$

3. Non-Zero Separation Constant Solutions: $k \neq 0$

$$\frac{d^2 X}{dx^2} + k^2 X = 0 \quad \Rightarrow \quad X = A_1 \sin kx + A_2 \cos kx$$

$$\frac{d^2 Y}{dy^2} - k^2 Y = 0 \quad \Rightarrow \quad Y = B_1 e^{ky} + B_2 e^{-ky}$$
$$= C_1 \sinh ky + D_1 \cosh ky$$

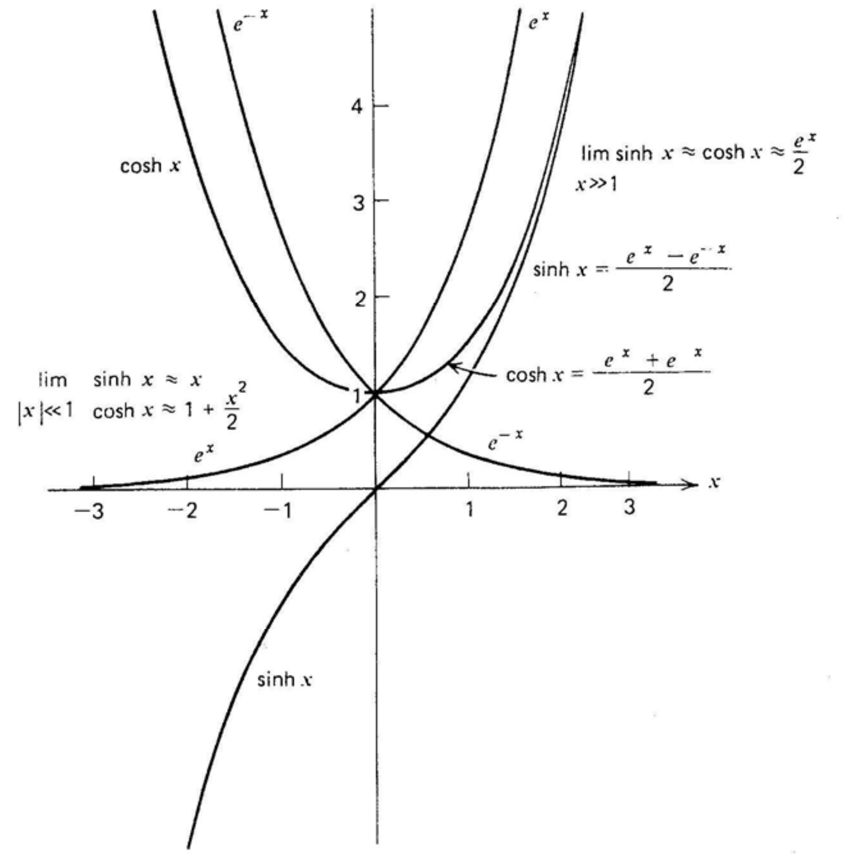


Figure 4-3 The exponential and hyperbolic functions for positive and negative arguments.

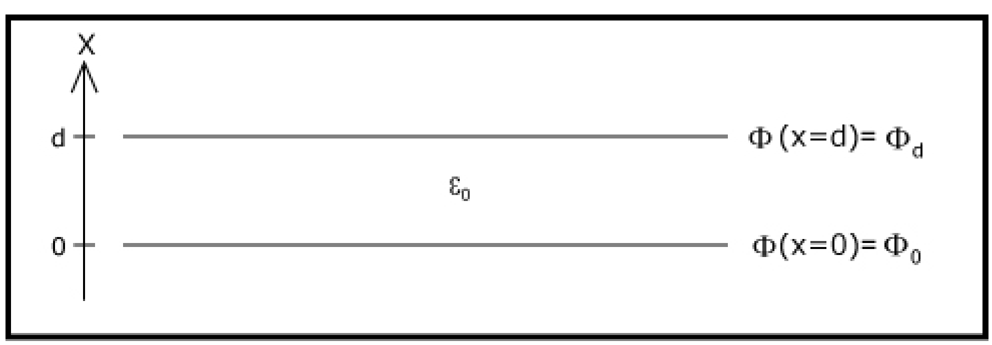
Courtesy of Krieger Publishing. Used with permission.

$$\Phi(x,y) = X(x)\,Y(y)$$

$$= D_1 \sin kx e^{ky} + D_2 \sin kx e^{-ky} + D_3 \cos kx e^{ky} + D_4 \cos kx e^{-ky}$$

$$= E_1 \sin kx \sinh ky + E_2 \sin kx \cosh ky + E_3 \cos kx \sinh ky + E_4 \cos kx \cosh ky$$

4. Parallel Plate Electrodes



Neglecting end effects, $\Phi(x)$. Boundary conditions are:

$$\Phi(x=0)=\Phi_0\ ,\ \ \Phi(x=d)=\Phi_d$$

Try zero separation constant solution:

$$\Phi(x)=a_1x+b_1$$

$$\Phi(x=0)=\Phi_0=b_1$$

$$\Phi(x=d)=\Phi_d=a_1d+b_1\ \ \Rightarrow\ \ a_1=\frac{\Phi_d-\Phi_0}{d}$$

$$\Phi(x)=\frac{\Phi_d-\Phi_0}{d}x+\Phi_0$$

$$E_x=-\frac{d\Phi}{dx}=\frac{\Phi_0-\Phi_d}{d}$$ (Electric field is uniform and equal to potential difference divided by spacing)

5. Hyperbolic Electrode Boundary Conditions

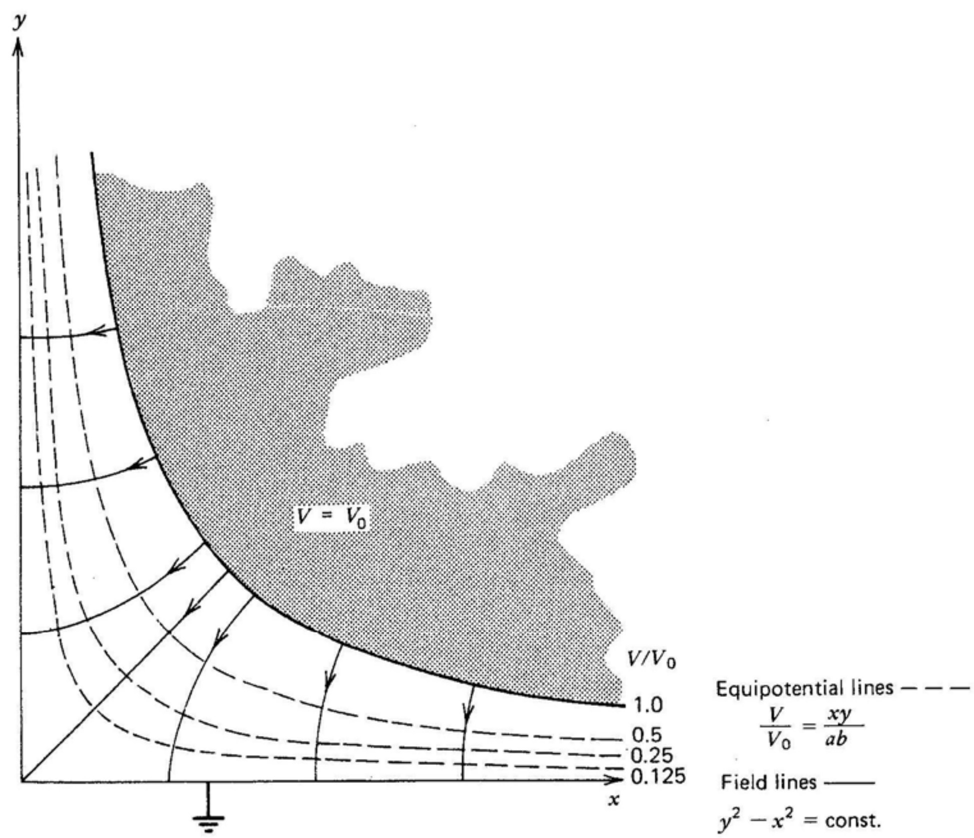


Figure 4-1 The equipotential and field lines for a hyperbolically shaped electrode at potential V_0 above a right-angle conducting corner are orthogonal hyperbolas.

$\Phi(xy = ab) = V_0$

$\Phi(x = 0, y) = 0$

$\Phi(x, y = 0) = 0$

$\Phi(x, y) = V_0 \, xy/(ab)$

$$\bar{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial x}\bar{i}_x - \frac{\partial\Phi}{\partial y}\bar{i}_y$$
$$= -\frac{V_0}{ab}\left[y\,\bar{i}_x + x\,\bar{i}_y\right]$$

Electric field lines:

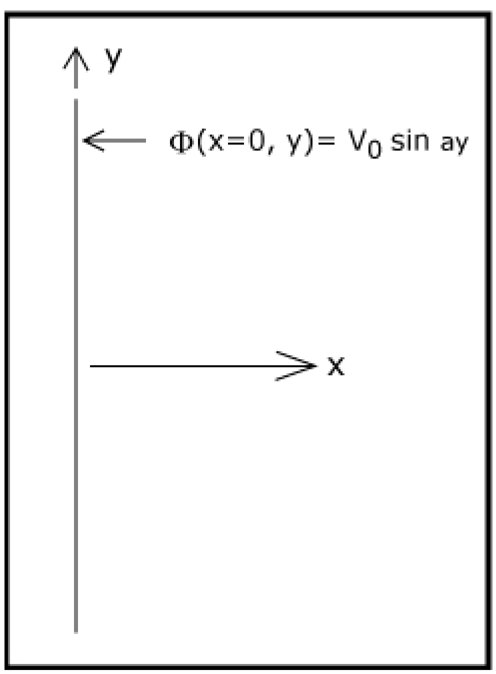
$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{x}{y}$$

$$ydy = xdx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$y^2 = x^2 + y_0^2 - x_0^2$ (field line passes through (x_0, y_0))

6. Spatially Periodic Potential Sheet



$$\Phi(x,y) = \begin{cases} V_0 \sin ay \, e^{-ax} & x \geq 0 \\ V_0 \sin ay \, e^{+ax} & x \leq 0 \end{cases}$$

$$\bar{E} = -\nabla\Phi(x,y) = -\left[\frac{\partial\Phi}{\partial x}\bar{i}_x + \frac{\partial\Phi}{\partial y}\bar{i}_y\right]$$

$$= \begin{cases} -V_0 a \, e^{-ax}\left[\cos ay \, \bar{i}_y - \sin ay \, \bar{i}_x\right] & x > 0 \\ -V_0 a \, e^{ax}\left[\cos ay \, \bar{i}_y + \sin ay \, \bar{i}_x\right] & x < 0 \end{cases}$$

$$\sigma_s(x=0) = \varepsilon_0\left[E_x(x=0_+) - E_x(x=0_-)\right] = 2\varepsilon_0 V_0 \, a \sin ay$$

7. Electric Field Lines:

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \begin{cases} -\cot ay & x > 0 \\ +\cot ay & x < 0 \end{cases}$$

$x > 0 \quad \cos ay \, e^{-ax} = \text{constant}$

$x < 0 \quad \cos ay \, e^{+ax} = \text{constant}$

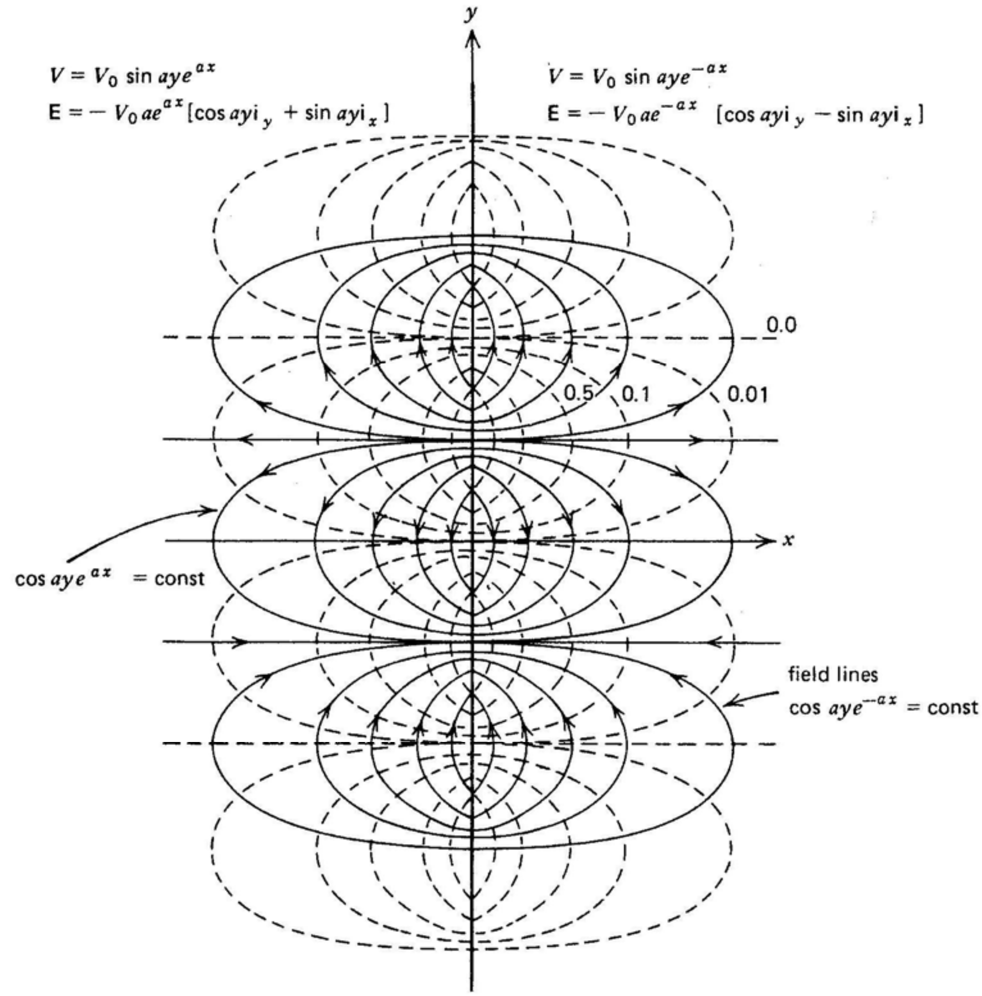


Figure 4-4 The potential and electric field decay away from an infinite sheet with imposed spatially periodic voltage. The field lines emanate from positive surface charge on the sheet and terminate on negative surface charge.

Courtesy of Krieger Publishing. Used with permission.

X. <u>Two-Dimensional Solutions to Laplace's Equation in Polar Coordinates</u> $\left(\frac{\partial}{\partial z} = 0\right)$

1. Product Solution

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \underbrace{\frac{\partial^2 \Phi}{\partial z^2}}_{0} = 0$$

$$\Phi(r, \phi) = R(r)\, F(\phi)$$

$\frac{V}{V_0}$ – – – – –

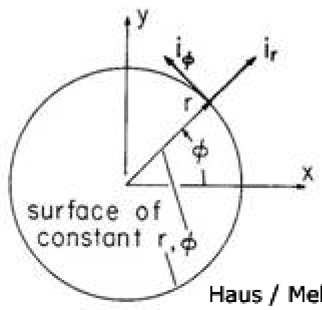
$$\frac{F(\phi)}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{R(r)}{r^2}\frac{d^2F}{d\phi^2}=0$$

Multiply through by $\frac{r^2}{R(r)F(\phi)}$

$$\underbrace{\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}_{m^2}+\underbrace{\frac{1}{F}\frac{d^2F}{d\phi^2}}_{-m^2}=0$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=m^2 \Rightarrow r\frac{d}{dr}\left(r\frac{dR}{dr}\right)-m^2R=0$$

$$\frac{1}{F}\frac{d^2F}{d\phi^2}=-m^2 \Rightarrow \frac{d^2F}{d\phi^2}+m^2F=0$$



Haus / Melcher **Figure 5.7.1** Polar coordinate system.

Courtesy of MIT Press. Used with permission.

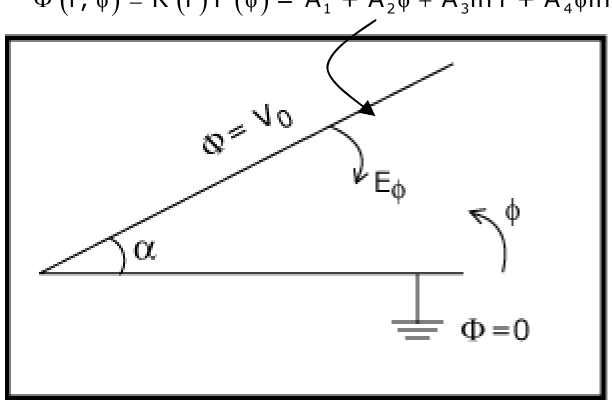
2. m=0 Solutions (Zero Separation Constant Solutions)

$$r\frac{dR}{dr}=C \Rightarrow R=C\ln r+D$$

$$\frac{d^2F}{d\phi^2}=0 \Rightarrow F=A\phi+B$$

Potential of line charge

$$\Phi(r,\phi)=R(r)F(\phi)=A_1+A_2\phi+A_3\ln r+A_4\phi\ln r$$



$$\left.\begin{array}{l}\Phi(\phi=0)=0\\ \Phi(\phi=\alpha)=V_0\end{array}\right\}\Rightarrow\ \Phi(\phi)=\frac{V_0\phi}{\alpha}\ \Rightarrow\ \bar{E}=-\nabla\Phi=-\left[\bar{i}_r\frac{\partial\Phi}{\partial r}+\bar{i}_\theta\frac{1}{r}\frac{\partial\Phi}{\partial\phi}+\bar{i}_z\frac{\partial\Phi}{\partial z}\right]$$

$$E_\phi=-\frac{1}{r}\frac{\partial\Phi}{\partial\phi}=-\frac{V_0}{\alpha r}$$

$$\sigma_s(r,\phi=0)=\varepsilon_0E_\phi(r,\phi=0)=-\frac{\varepsilon_0V_0}{\alpha r}$$

$$\sigma_s(r,\phi=\alpha)=-\varepsilon_0E_\phi(r,\phi=0)=+\frac{\varepsilon_0V_0}{\alpha r}$$

3. $m\neq0$ Solutions (Non-Zero Separation Constant Solutions)

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)-m^2R=0$$

Try $R=Ar^n$

$$r\frac{d}{dr}\left[nAr^n\right]-m^2Ar^n=0$$

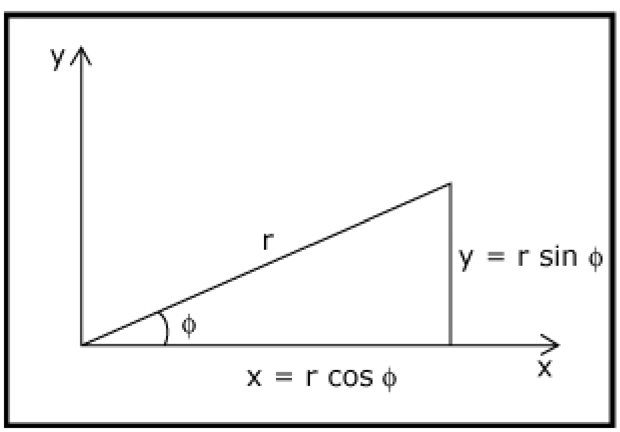
$$n^2r^n-m^2r^n=0\ \Rightarrow\ n=\pm m$$

$$R(r)=A_3r^m+A_4r^{-m}$$

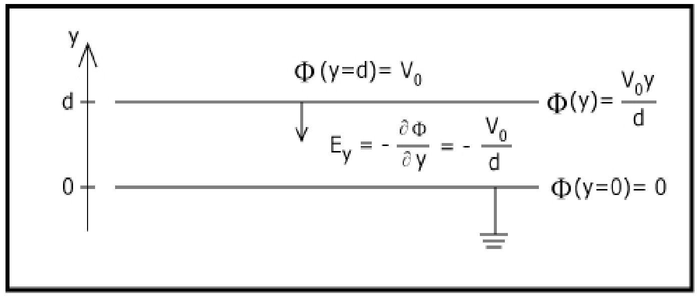
$$\frac{d^2F}{d\phi^2}+m^2F=0$$

$$F=A_1\sin m\phi+A_2\cos m\phi$$

$$\Phi(r,\phi)=R(r)F(\phi)=\left[A_1\sin m\phi+A_2\cos m\phi\right]\left[A_3r^m+A_4r^{-m}\right]$$
$$=A\sin m\phi r^m+B\sin m\phi r^{-m}+C\cos m\phi r^m+D\cos m\phi r^{-m}$$



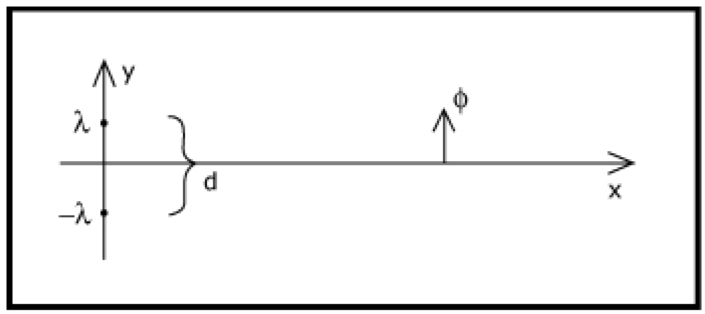
4. Selected Solutions



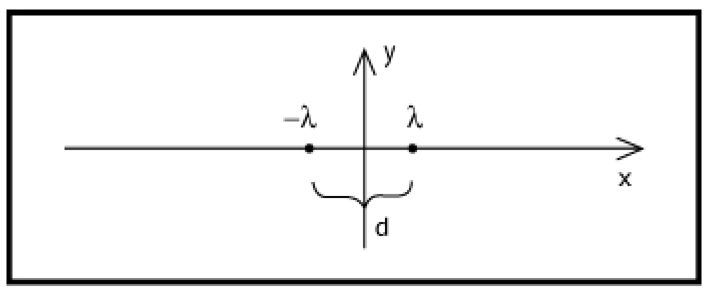
m=1

$\Phi(r, \phi) = Ar \sin \phi = Ay$

$\Phi(r, \phi) = Cr \cos \phi = Cx$



$\Phi(r, \phi) = \dfrac{B \sin \phi}{r} \Rightarrow$ Line dipole oriented in y direction



$\Phi(r, \phi) = \dfrac{D \cos \phi}{r} \Rightarrow$
Line dipole oriented in x direction

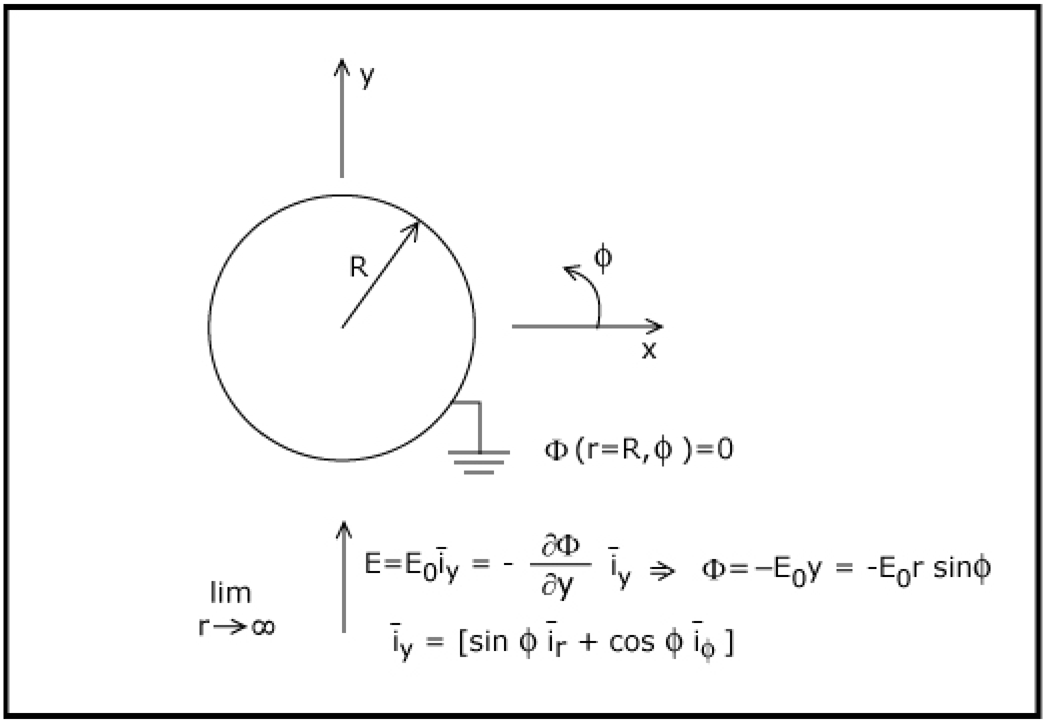
m=2

$\Phi = Ar^2 \sin 2\phi = 2Ar^2 \sin\phi \cos \phi = 2Axy$

Generally m an integer if $\Phi(\phi = 0) = \Phi(\phi = 2\pi)$

5. Grounded Perfectly Conducting Cylinder in a Uniform y Directed Electric Field

$\Phi = \left[-E_0 r + \dfrac{A}{r}\right] \sin \phi \qquad r \geq R$



$$\Phi\left(r = R, \phi\right) = 0 \;\Rightarrow\; -E_0R + \frac{A}{R} = 0 \;\Rightarrow\; A = E_0R^2$$

$$\Phi = -E_0\left(r - \frac{R^2}{r}\right)\sin\phi \qquad r \geq R$$

$$\bar{E} = -\nabla\Phi = -\left[\frac{\partial\Phi}{\partial r}\bar{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\bar{i}_\Phi + \cancelto{0}{\frac{\partial\Phi}{\partial z}\bar{i}_z}\right]$$

$$= E_0\left\{\left[1 + \frac{R^2}{r^2}\right]\sin\phi\,\bar{i}_r - \left[1 - \frac{R^2}{r^2}\right]\cos\phi\,\bar{i}_\Phi\right\} \qquad r > R$$

$$\sigma_s\left(r = R, \phi\right) = \varepsilon_0\left[E_r\left(r = R_+, \phi\right) - \cancelto{0}{E_r\left(r = R_-, \phi\right)}\right] = 2\varepsilon_0E_0\sin\phi$$

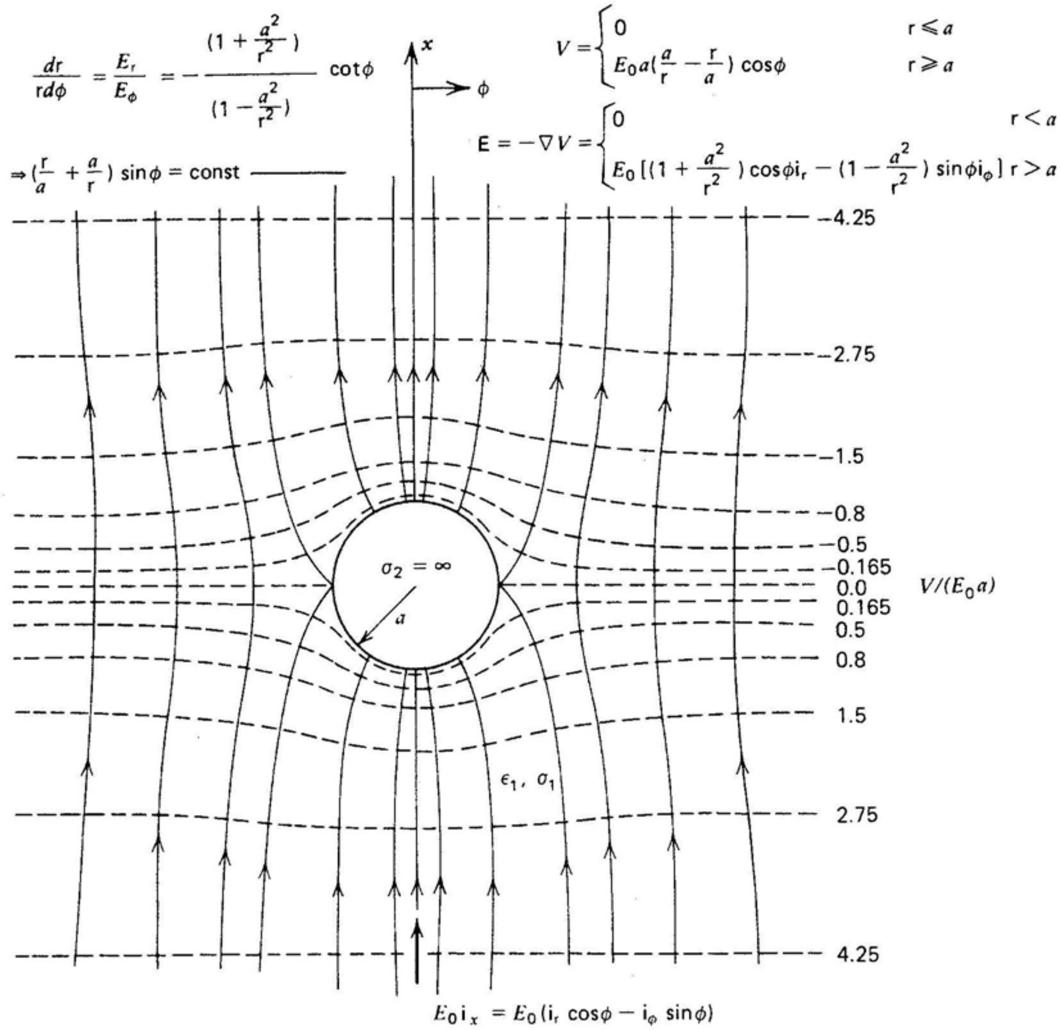


Figure 4-8 Steady-state field and equipotential lines about a perfectly conducting cylinder in a uniform electric field.

Courtesy of Krieger Publishing. Used with permission.

XI. Two-Dimensional Solutions to Laplace's Equation in Spherical Coordinates $\left(\frac{\partial}{\partial \phi} = 0\right)$

1. Product Solution

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \underbrace{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}}_{0} = 0$$

$$\Phi(r, \theta) = R(r)\,\Theta(\theta)$$

$$\frac{\Theta(\theta)}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{R}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = 0$$

Multiply through by $\dfrac{r^2}{R(r)\,\Theta(\theta)}$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{n(n+1)}+\underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{-n(n+1)}=0$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-n(n+1)\,R=0$$

Try $R = Ar^p$

$$\not{A}p(p+1)\not{r^p}-n(n+1)\not{A}\not{r^p}=0 \Rightarrow p=n,-(n+1)$$

$$R(r)=Ar^n+Br^{-(n+1)}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+n(n+1)\sin\theta\Theta=0$$ [Legendre's Equation]

In this course, only responsible for n=1 solution

$$\Rightarrow \Theta(\theta)=\cos\theta$$

$\Phi = Ar\cos\theta = Az$ is potential of uniform z directed electric field

$\Phi = \dfrac{B\cos\theta}{r^2}$ is potential of point electric dipole

n=1

$$\Phi(r,\theta)=\left(Ar+\frac{B}{r^2}\right)\cos\theta$$

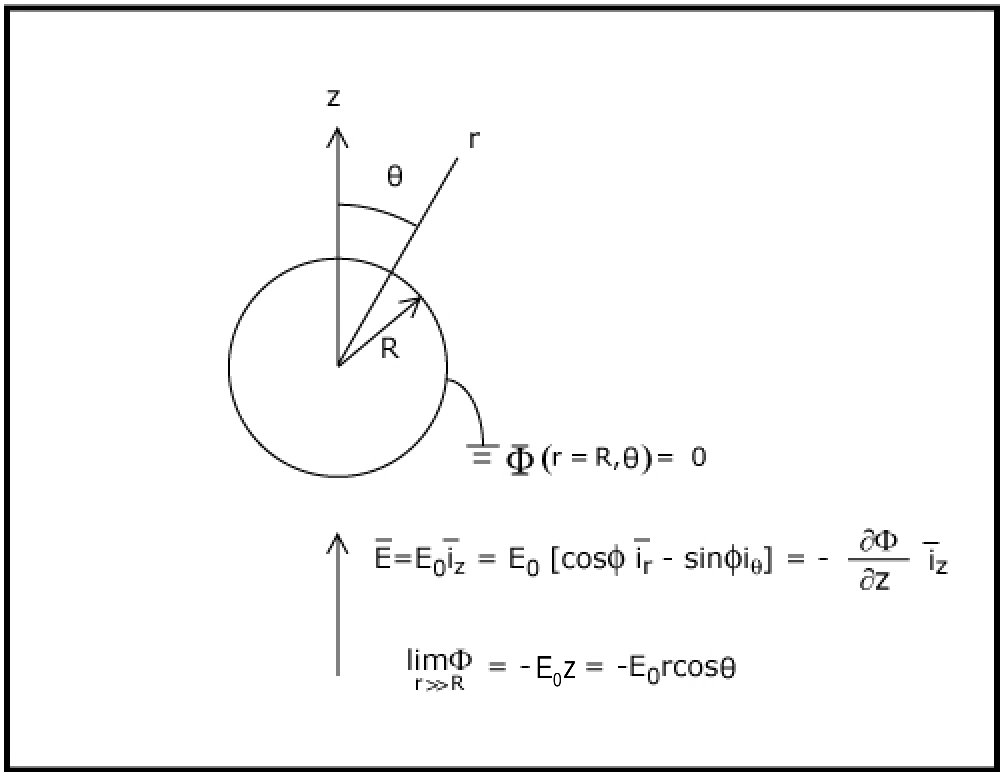
2. Grounded Sphere in a Uniform z Directed Electric Field

$$z = r\cos\theta$$

$$\nabla z=\bar{i}_z=\nabla(r\cos\theta)=\bar{i}_r\frac{\partial}{\partial r}(r\cos\theta)+\bar{i}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(r\cos\theta)=\bar{i}_r\cos\theta-\bar{i}_\theta\sin\theta$$

$r \geq R$

$$\Phi(r,\theta)=-E_0r\cos\theta+\frac{A\cos\theta}{r^2}$$



$$\Phi(r = R, \theta) = 0 = \left(-E_0 R + \frac{A}{R^2}\right)\cos\theta \Rightarrow A = E_0 R^3$$

$$\Phi(r, \theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta \qquad r \geq R$$

$$\bar{E} = -\nabla\Phi = -\left[\frac{\partial \Phi}{\partial r}\bar{i}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\bar{i}_\theta\right]$$

$$= E_0\left[\left(1 + \frac{2R^3}{r^3}\right)\cos\theta\ \bar{i}_r - \left(1 - \frac{R^3}{r^3}\right)\sin\theta\ \bar{i}_\theta\right]$$

$$\sigma_s(r = R, \theta) = \varepsilon_0 E_r(r = R, \theta) = 3\varepsilon_0 E_0 \cos\theta$$

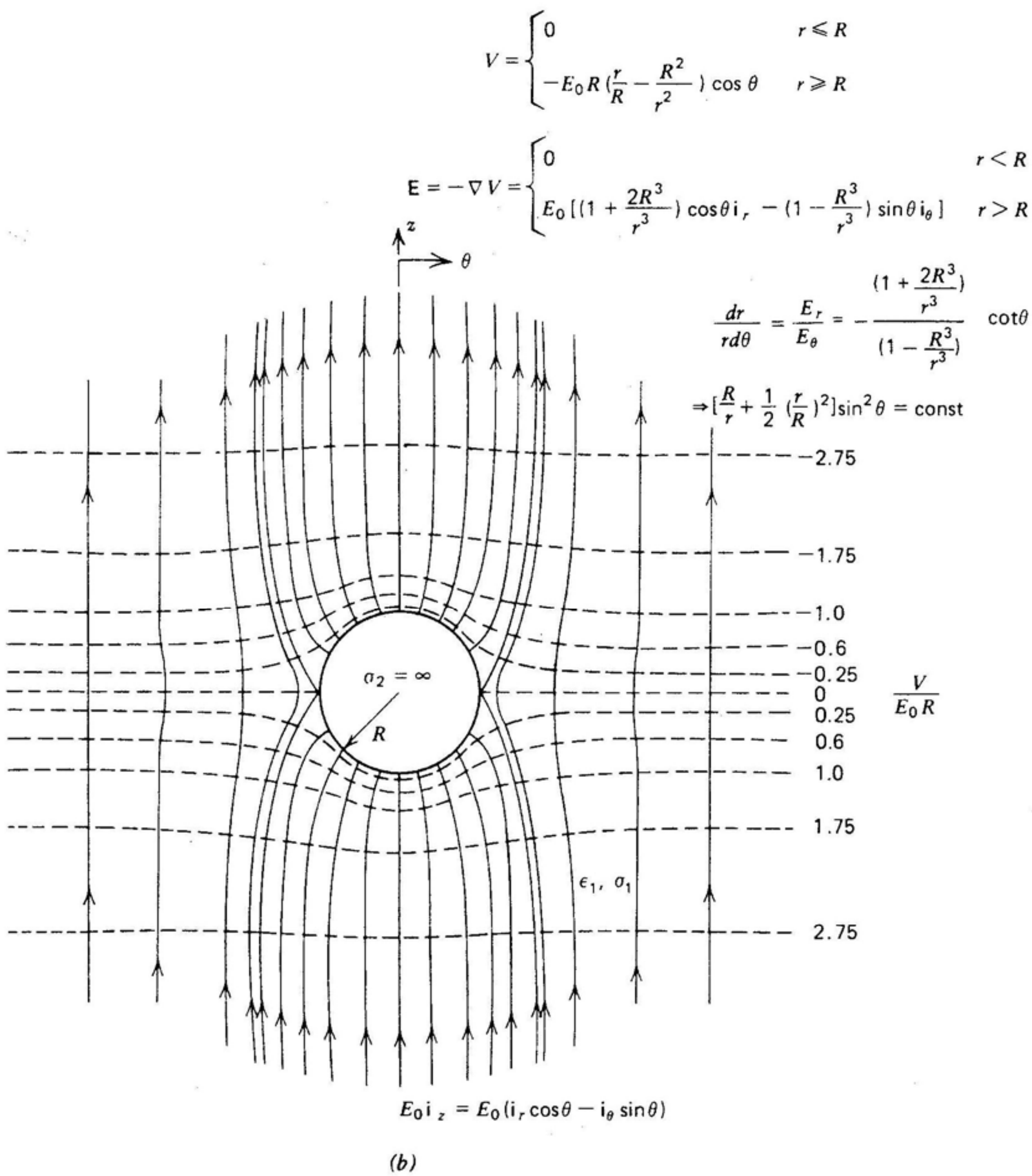

$V = \begin{cases} 0 & r \leqslant R \\ -E_0 R \left(\frac{r}{R} - \frac{R^2}{r^2}\right) \cos\theta & r \geqslant R \end{cases}$
$\mathbf{E} = -\nabla V = \begin{cases} 0 & r < R \\ E_0 \left[\left(1 + \frac{2R^3}{r^3}\right) \cos\theta \, \mathbf{i}_r - \left(1 - \frac{R^3}{r^3}\right) \sin\theta \, \mathbf{i}_\theta\right] & r > R \end{cases}$
z
θ
$\frac{dr}{r d\theta} = \frac{E_r}{E_\theta} = -\frac{\left(1 + \frac{2R^3}{r^3}\right)}{\left(1 - \frac{R^3}{r^3}\right)} \cot\theta$
$\Rightarrow \left[\frac{R}{r} + \frac{1}{2}\left(\frac{r}{R}\right)^2\right] \sin^2\theta = \text{const}$
−2.75
−1.75
−1.0
−0.6
−0.25
0
0.25
0.6
1.0
1.75
2.75
$\frac{V}{E_0 R}$
$\sigma_2 = \infty$
R
ϵ_1, σ_1
$E_0 \mathbf{i}_z = E_0 (\mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta)$
(b)

Figure 4-12b