6.642 Continuum Electromechanics
Fall 2008

6.642, Continuum Electromechanics

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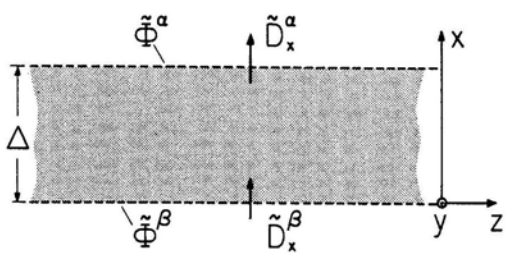
Lecture 2: Continuum Electromechanics (Melcher) – Section 2.16
Flux-Potential Relations for Laplacian Fields

I. Planar Layers

A. Electric Fields

Flux-potential transfer relations for planar layer in terms of electric potential and normal displacement (Φ, D_x). To obtain magnetic relations, substitute $(\Phi, D_x, \varepsilon) \rightarrow (\Psi, B_x, \mu)$.

Planar layer



$$\begin{bmatrix} \tilde{D}_x^\alpha \\ \tilde{D}_x^\beta \end{bmatrix} = \varepsilon\gamma \begin{bmatrix} -\coth(\gamma\Delta) & \frac{1}{\sinh(\gamma\Delta)} \\ \frac{-1}{\sinh(\gamma\Delta)} & \coth(\gamma\Delta) \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \quad \text{(a)}$$

$$\begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} = \frac{1}{\varepsilon\gamma} \begin{bmatrix} -\coth(\gamma\Delta) & \frac{1}{\sinh(\gamma\Delta)} \\ \frac{-1}{\sinh(\gamma\Delta)} & \coth(\gamma\Delta) \end{bmatrix} \begin{bmatrix} \tilde{D}_x^\alpha \\ \tilde{D}_x^\beta \end{bmatrix} \quad \text{(b)}$$

$$\Phi = \text{Re}\, \tilde{\Phi}(x,t)\, e^{-j(k_y y + k_z z)}$$

$$\gamma \equiv \sqrt{k_y^2 + k_z^2}$$

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$$\nabla \cdot \bar{E} = 0$$

$$\nabla \times \bar{E} = 0 \Rightarrow \bar{E} = -\nabla \Phi$$

$$\nabla^2 \Phi = 0$$

$$\Phi(x, y, z, t) = \text{Re}\left[\tilde{\Phi}(x, t)\, e^{-j(k_y y + k_z z)}\right]$$

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\frac{d^2 \tilde{\Phi}}{dx^2} - \left(k_y^2 + k_z^2\right)\tilde{\Phi} = 0 \quad ; \quad \gamma = \sqrt{k_y^2 + k_z^2}$$

$$\tilde{\Phi} = \tilde{\Phi}_1 \sinh \gamma x + \tilde{\Phi}_2 \cosh \gamma x$$

$$\tilde{\Phi}(x = \Delta) = \tilde{\Phi}^\alpha \quad ; \quad \tilde{\Phi}(x = 0) = \tilde{\Phi}^\beta$$

$$\tilde{\Phi}(x) = \frac{\tilde{\Phi}^{\alpha} \sinh \gamma x - \tilde{\Phi}^{\beta} \sinh \gamma (x - \Delta)}{\sinh \gamma\Delta}$$

$$D_x = -\varepsilon \frac{\partial \Phi}{\partial x} \Rightarrow \tilde{D}_x = \frac{-\varepsilon\gamma \left[\tilde{\Phi}^{\alpha} \cosh \gamma x - \tilde{\Phi}^{\beta} \cosh \gamma (x - \Delta)\right]}{\sinh \gamma\Delta}$$

$$\tilde{D}_x^{\alpha} = \tilde{D}_x (x = \Delta) = \frac{-\varepsilon\gamma}{\sinh \gamma\Delta} \left[\tilde{\Phi}^{\alpha} \cosh \gamma\Delta - \tilde{\Phi}^{\beta}\right]$$

$$\tilde{D}_x^{\beta} = \tilde{D}_x (x = 0) = \frac{-\varepsilon\gamma}{\sinh \gamma\Delta} \left[\tilde{\Phi}^{\alpha} - \tilde{\Phi}^{\beta} \cosh \gamma\Delta\right]$$

B. Magnetic Fields

$$\nabla \times \bar{H} = 0 \Rightarrow \bar{H} = -\nabla \Psi$$

$$\nabla \bullet \bar{H} = 0 \Rightarrow \nabla^2 \Psi = 0 = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2}$$

$$\Psi (x, y, z, t) = \mathrm{Re}\left[\tilde{\Psi}(x, t) e^{-j(k_y y + k_z z)}\right]$$

$$\frac{d^2 \tilde{\Psi}}{dx^2} - \gamma^2 \tilde{\Psi} = 0 \quad ; \quad \gamma = \sqrt{k_y^2 + k_z^2}$$

$$\tilde{\Psi}(x) = \frac{\tilde{\Psi}^{\alpha} \sinh \gamma x - \tilde{\Psi}^{\beta} \sinh \gamma (x - \Delta)}{\sinh \gamma\Delta}$$

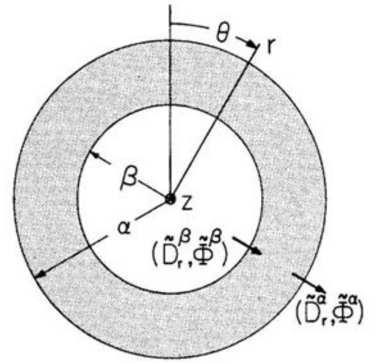
$$B_x = -\mu \frac{\partial \Psi}{\partial x} \Rightarrow \tilde{B}_x (x) = \frac{-\mu\gamma \left[\tilde{\Psi}^{\alpha} \cosh \gamma x - \tilde{\Psi}^{\beta} \cosh \gamma (x - \Delta)\right]}{\sinh \gamma\Delta}$$

$$\tilde{B}_x^{\alpha} = \tilde{B}_x (x = \Delta) = \frac{-\mu\gamma}{\sinh \gamma\Delta} \left[\tilde{\Psi}^{\alpha} \cosh \gamma\Delta - \tilde{\Psi}^{\beta}\right]$$

$$\tilde{B}_x^{\beta} = \tilde{B}_x (x = 0) = \frac{-\mu\gamma}{\sinh \gamma\Delta} \left[\tilde{\Psi}^{\alpha} - \tilde{\Psi}^{\beta} \cosh \gamma\Delta\right]$$

II. Cylindrical Annulus

Flux-potential relations for cylindrical annulus in terms of electric potential and normal displacement (Φ, D_r). To obtain magnetic relations, substitute $(\Phi, D_r, \varepsilon) \rightarrow (\Psi, B_r, \mu)$.



$$\Phi = \text{Re}\, \tilde{\Phi}(r,t) e^{-j(m\theta + kz)}$$

$$\begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} = \varepsilon \begin{bmatrix} f_m(\beta,\alpha) & g_m(\alpha,\beta) \\ g_m(\beta,\alpha) & f_m(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \quad \text{(a)}$$

$$\begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} F_m(\beta,\alpha) & G_m(\alpha,\beta) \\ G_m(\beta,\alpha) & F_m(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} \quad \text{(b)}$$

<u>$k = 0,\ m = 0$</u>

(a): $f_o(x,y) = \frac{1}{y} / \ln\left(\frac{x}{y}\right); \quad g_o(x,y) = \frac{1}{x} / \ln\left(\frac{x}{y}\right)$

(b): No inverse

<u>$k = 0,\ m = 1,2,\cdots$</u>

(a):

$$f_m(x,y) = \frac{m}{y} \frac{\left[\left(\frac{x}{y}\right)^m + \left(\frac{y}{x}\right)^m\right]}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

$$g_m(x,y) = \frac{2m}{x} \frac{1}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

(b):

$$F_m(x,y) = \frac{y}{m} \frac{\left[\left(\frac{x}{y}\right)^m + \left(\frac{y}{x}\right)^m\right]}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

$$G_m(x,y) = \frac{2y}{m} \frac{1}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

<u>$k \neq 0,\ m = 0,1,2\cdots$*</u>

(a):

$$f_m(x,y) = \frac{jk[H_m(jkx)J_m'(jky) - J_m(jkx)H_m'(jky)]}{[J_m(jkx)H_m(jky) - J_m(jky)H_m(jkx)]}$$

$$g_m(x,y) = \frac{-2j}{\pi x[J_m(jkx)H_m(jky) - J_m(jky)H_m(jkx)]}$$

$$f_m(x,y) = \frac{k[K_m(kx)I_m'(ky) - I_m(kx)K_m'(ky)]}{[I_m(kx)K_m(ky) - I_m(ky)K_m(kx)]}$$

$$g_m(x,y) = \frac{1}{x[I_m(kx)K_m(ky) - I_m(ky)K_m(kx)]}$$

(b):

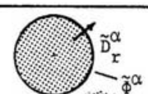
$$F_m(x,y) = \frac{1}{jk} \frac{[J_m'(jkx)H_m(jky) - H_m'(jkx)J_m(jky)]}{[J_m'(jky)H_m'(jkx) - J_m'(jkx)H_m'(jky)]}$$

$$G_m(x,y) = \frac{-2}{j\pi k(kx)[J_m'(jky)H_m'(jkx) - J_m'(jkx)H_m'(jky)]}$$

$$F_m(x,y) = \frac{1}{k} \frac{[I_m'(kx)K_m(ky) - K_m'(kx)I_m(ky)]}{[I_m'(ky)K_m'(kx) - I_m'(kx)K_m'(ky)]}$$

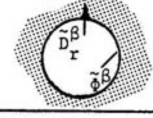
$$G_m(x,y) = \frac{1}{k(kx)[I_m'(ky)K_m'(kx) - I_m'(kx)K_m'(ky)]}$$

$\beta \rightarrow 0$



$$\tilde{D}_r^\alpha = \varepsilon f_m(0,\alpha)\tilde{\Phi}^\alpha; \quad f_m(0,\alpha) = -\frac{kI_m'(k\alpha)}{I_m(k\alpha)} \quad \text{(c)}$$

$\alpha \rightarrow \infty$



$$\tilde{D}_r^\beta = \varepsilon f_m(\infty,\beta)\tilde{\Phi}^\beta; \quad f_m(\infty,\beta) = -\frac{kK_m'(k\beta)}{K_m(k\beta)} \quad \text{(d)}$$

*See Prob. 2.17.2 for proof that $H_m(jkx)J_m'(jkx) - J_m(jkx)H'_m(jkx) = -2/(\pi kx)$ and $K_m(kx)I_m'(kx) - I_m(kx)K_m'(kx) = 1/kx$ incorporated into g_m and G_m.

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\Phi(r,\theta,z,t) = \text{Re}\left[\tilde{\Phi}(r,t)e^{-j(m\theta+kz)}\right]$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\tilde{\Phi}}{dr}\right) - \frac{m^2}{r^2}\tilde{\Phi} - k^2\tilde{\Phi} = 0$$

1. m=0, k=0 solutions

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\tilde{\Phi}}{dr}\right) = 0 \Rightarrow r\frac{d\tilde{\Phi}}{dr} = C \Rightarrow \tilde{\Phi} = C\ln r + D$$

$$\tilde{\Phi}(r) = \frac{\tilde{\Phi}^\alpha \ln\frac{r}{\beta} - \tilde{\Phi}^\beta \ln\frac{r}{\alpha}}{\ln\frac{\alpha}{\beta}}$$

$$D_r = -\varepsilon\frac{\partial\Phi}{\partial r} \Rightarrow \tilde{D}_r = -\varepsilon\frac{\tilde{\Phi}^\alpha - \tilde{\Phi}^\beta}{r\ln\frac{\alpha}{\beta}}$$

$$\tilde{D}_r^\alpha = -\varepsilon\frac{\tilde{\Phi}^\alpha - \tilde{\Phi}^\beta}{\alpha\ln\frac{\alpha}{\beta}}$$

$$\tilde{D}_r^\beta = -\varepsilon\frac{\tilde{\Phi}^\alpha - \tilde{\Phi}^\beta}{\beta\ln\frac{\alpha}{\beta}}$$

$$\begin{bmatrix}\tilde{D}_r^\alpha \\ \tilde{D}_r^\beta\end{bmatrix} = \frac{-\varepsilon}{\ln\frac{\alpha}{\beta}}\begin{bmatrix}\frac{1}{\alpha} & -\frac{1}{\alpha} \\ \frac{1}{\beta} & -\frac{1}{\beta}\end{bmatrix}\begin{bmatrix}\tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta\end{bmatrix}$$

2. k=0, m≠0

$$r\frac{d}{dr}\left(r\frac{d\tilde{\Phi}}{dr}\right) - m^2\tilde{\Phi} = 0 \Rightarrow \tilde{\Phi} = A_1 r^m + A_2 r^{-m}$$

$$\tilde{\Phi} = \tilde{\Phi}^\alpha\frac{\left[\left(\frac{\beta}{r}\right)^m - \left(\frac{r}{\beta}\right)^m\right]}{\left[\left(\frac{\beta}{\alpha}\right)^m - \left(\frac{\alpha}{\beta}\right)^m\right]} + \tilde{\Phi}^\beta\frac{\left[\left(\frac{r}{\alpha}\right)^m - \left(\frac{\alpha}{r}\right)^m\right]}{\left[\left(\frac{\beta}{\alpha}\right)^m - \left(\frac{\alpha}{\beta}\right)^m\right]}$$

3. $k \neq 0$, $m \neq 0$

$$\tilde{\Phi} = A_1\, I_m(kr) + A_2\, K_m(kr) \quad \text{[Modified Bessel Functions]}$$

$$I_m(r \to \infty) = \infty\ ,\ \ I_m(r \to 0) \to \text{finite}$$

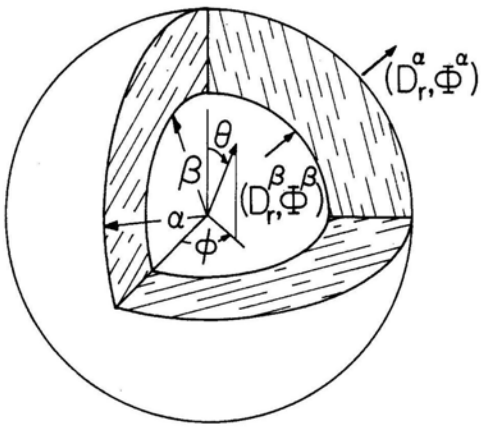
$$K_m(r \to \infty) \to 0\ ,\ \ K_m(r \to 0) \to \infty$$

$$J_m(jkr) = j^m I_m(kr)\,,\ H_m(jkr) = \frac{2}{\pi} j^{-(m+1)} K_m(kr)$$

$$\tilde{\Phi}(r) = \frac{\left\{\tilde{\Phi}^{\alpha}\left[H_m(jk\beta)J_m(jkr) - J_m(jk\beta)H_m(jkr)\right] + \tilde{\Phi}^{\beta}\left[J_m(jk\alpha)H_m(jkr) - H_m(jk\alpha)J_m(jkr)\right]\right\}}{\left[H_m(jk\beta)J_m(jk\alpha) - J_m(jk\beta)H_m(jk\alpha)\right]}$$

III. Spherical Shell

Flux-potential transfer relations for spherical shell in terms of electric potential and normal displacement (Φ, D_r). To obtain magnetic relations, substitute $(\Phi, D_r, \varepsilon) \rightarrow (\Psi, B_r, \mu)$.



$$\Phi = \mathrm{Re}\, \tilde{\Phi}(r,t) P_n^m(\cos\theta) e^{-jm\phi}$$

$$P_n^m = (1 - x^2)^{m/2} \frac{d^m P_n}{dx^m}$$

$$P_o = 1,\; P_1 = x,\; P_2 = \tfrac{1}{2}(3x^2 - 1)$$

$$P_3 = \tfrac{1}{2}(5x^3 - 3x)$$

$$P_4 = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$

m	P_o^m	P_1^m	$P_1^m \cos m\phi$	P_2^m	$P_2^m \cos m\phi$	P_3^m	$P_3^m \cos m\phi$
0	1	$\cos\theta$	+ / −	$\frac{1}{2}(3\cos^2\theta - 1)$	+ / − / +	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$	+ / − / + / −
1	0	$\sin\theta$	+ − +	$3\sin\theta\cos\theta$	+ − + / − + −	$\frac{3}{2}\sin\theta(5\cos^2\theta - 1)$	+ − + / − + − / + − +
2	0	0		$3\sin^2\theta$	+ − + − +	$15\sin^2\theta\cos\theta$	+ − + − + / − + − + −
3	0	0		0		$15\sin^3\theta$	+ − + − + − +

$$\begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} = \varepsilon \begin{bmatrix} f_n(\beta,\alpha) & g_n(\alpha,\beta) \\ g_n(\beta,\alpha) & f_n(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \quad \text{(a)}$$

$$f_n(x,y) = \frac{[n(\frac{y}{x})^n + (n+1)(\frac{x}{y})^{n+1}]}{[x(\frac{x}{y})^n - y(\frac{y}{x})^n]}$$

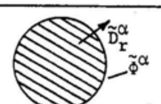
$$g_n(x,y) = \frac{(2n+1)}{x^2[\frac{1}{y}(\frac{x}{y})^n - \frac{1}{x}(\frac{y}{x})^n]}$$

$$\begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} F_n(\beta,\alpha) & G_n(\alpha,\beta) \\ G_n(\beta,\alpha) & F_n(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} \quad \text{(b)}$$

$$F_n(x,y) = \frac{y}{x} \frac{[\frac{1}{n}(\frac{y}{x})^n + \frac{1}{n+1}(\frac{x}{y})^{n+1}]}{[\frac{1}{y}(\frac{x}{y})^n - \frac{1}{x}(\frac{y}{x})^n]}$$

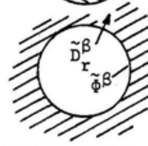
$$G_n(x,y) = \frac{y}{x} \frac{(2n+1)}{n(n+1)} \frac{1}{[\frac{1}{y}(\frac{x}{y})^n - \frac{1}{x}(\frac{y}{x})^n]}$$

$\beta \rightarrow 0$



$$\tilde{D}_r^\alpha = -\frac{\varepsilon n}{\alpha} \tilde{\Phi}^\alpha \quad \text{(c)}$$

$\alpha \rightarrow \infty$



$$\tilde{D}_r^\beta = \frac{\varepsilon(n+1)}{\beta} \tilde{\Phi}^\beta \quad \text{(d)}$$

$$\Phi(r,\theta,\phi) = \mathrm{Re}\left[\tilde{\Phi}(r)\Theta(\theta)e^{-jm\phi}\right]$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\frac{1}{\sin\theta\,\Theta}\frac{d}{d\theta}\left[\sin\theta\frac{\partial\Theta}{\partial\theta}\right] - \frac{m^2}{\sin^2\theta} = -k^2$$

$$\frac{1}{\tilde{\Phi}}\frac{d}{dr}\left(r^2\frac{d\tilde{\Phi}}{dr}\right) = k^2$$

$$u = \cos\theta,\ \sqrt{1-u^2} = \sin\theta$$

$$\left(1-u^2\right)\frac{d^2\Theta}{du^2} - 2u\frac{d\Theta}{du} + \left(k^2 - \frac{m^2}{1-u^2}\right)\Theta = 0$$

$k^2 = n\left(n+1\right)$, n an integer

$\Theta = P_n^m\left(u\right)$

$$\tilde{\Phi} = \tilde{\Phi}^{\alpha}\frac{\left[\left(\frac{r}{\beta}\right)^n - \left(\frac{\beta}{r}\right)^{n+1}\right]}{\left[\left(\frac{\alpha}{\beta}\right)^n - \left(\frac{\beta}{\alpha}\right)^{n+1}\right]} + \tilde{\Phi}^{\beta}\frac{\left[\left(\frac{r}{\alpha}\right)^n - \left(\frac{\alpha}{r}\right)^{n+1}\right]}{\left[\left(\frac{\beta}{\alpha}\right)^n - \left(\frac{\alpha}{\beta}\right)^{n+1}\right]}$$

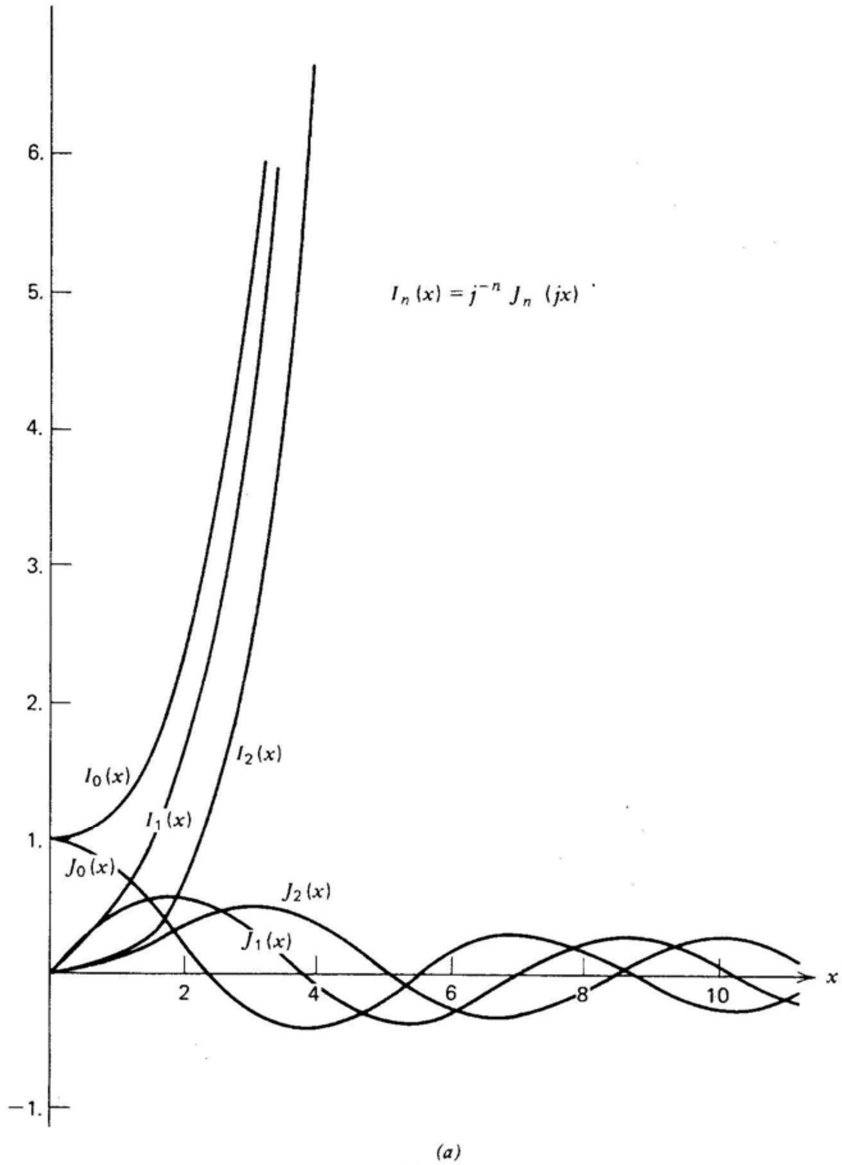


Figure 4-9a The Bessel functions (a) $J_n(x)$ and $I_n(x)$, and (b) $Y_n(x)$ and $K_n(x)$.

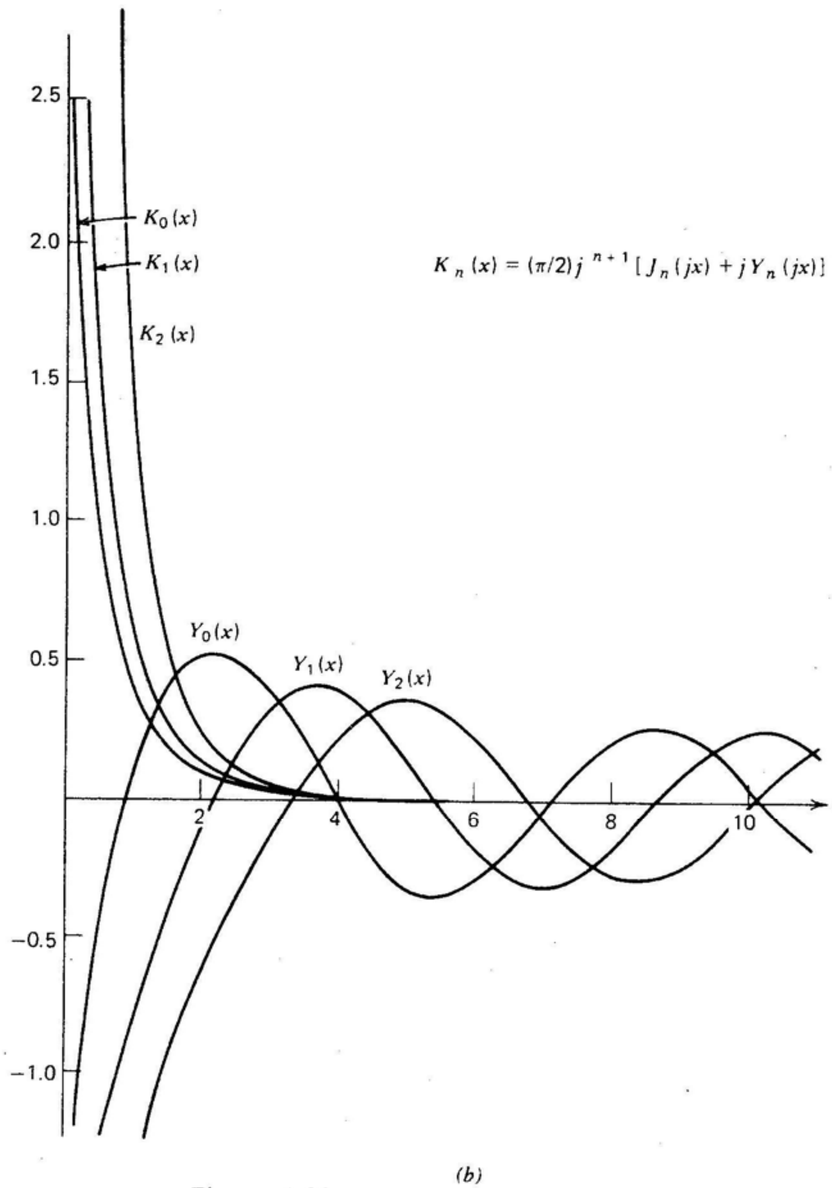


Figure 4-9*b*

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