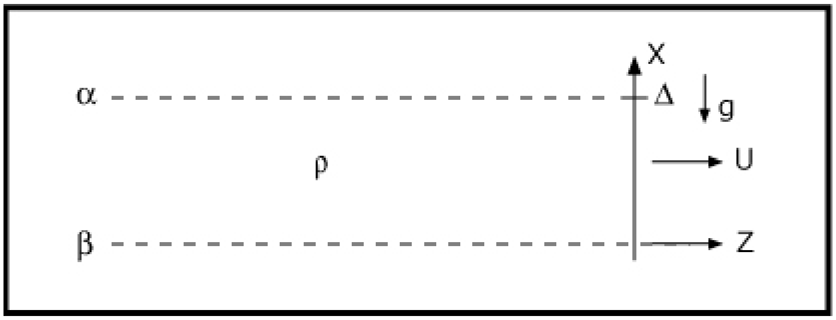
6.642 Continuum Electromechanics
Fall 2008

Lecture 7: Pressure–Velocity Relations for Inviscid, Incompressible Fluids

Continuum Electromechanics (Melcher) – Section 7.9, 8.9

I. Governing Equations



$$\rho\left(\frac{\partial \bar{v}}{\partial t}+\left(\bar{v}\cdot\nabla\right)\bar{v}\right)+\nabla p=\nabla\left(\rho\bar{g}\cdot\bar{r}-\mathcal{E}\right)$$

$$\nabla\cdot\bar{v}=0$$

Linearize:

$$\bar{v}=U\bar{i}_z+\bar{v}'$$

$$P=P_0\left(x\right)+p'$$

$$\mathcal{E}=\mathcal{E}_0\left(x\right)+\mathcal{E}'$$

$$\rho\frac{\partial \bar{v}'}{\partial t}+\rho U\frac{\partial}{\partial z}\bar{v}'+\nabla p'=-\nabla\mathcal{E}'$$

Take curl of Equation: $\Rightarrow\ \nabla\times\left(\nabla\left(p'+\mathcal{E}'\right)\right)=0\Rightarrow\nabla\times\bar{v}'=0$

$$\bar{v}'=-\nabla\Theta'$$

$\nabla\cdot\bar{v}'=0\Rightarrow\nabla^2\Theta'=0$ (Laplace's Equation)

$$\Theta'=\text{Re}\left[\hat{\Theta}\left(x\right)e^{j\left(\omega t-k_y y-k_z z\right)},\ k^2=k_y^2+k_z^2\right]$$

$$\hat{\Theta}\left(\Delta\right)=\hat{\Theta}^{\alpha}\qquad,\ \hat{v}_x\left(\Delta\right)=\hat{v}_x^{\alpha}$$

$$\hat{\Theta}'(0) = \hat{\Theta}^{\beta} \qquad , \ \hat{v}_x(0) = \hat{v}_x^{\beta}$$

$$\hat{\Theta}(x) = \frac{\hat{\Theta}^{\alpha} \sinh kx - \hat{\Theta}^{\beta} \sinh k(x - \Delta)}{\sinh k\Delta}$$

$$\hat{v}_x = -\frac{d\hat{\Theta}}{dx} = -\frac{k}{\sinh k\Delta}\left[\hat{\Theta}^{\alpha} \cosh kx - \hat{\Theta}^{\beta} \cosh k(x - \Delta)\right]$$

$$\hat{v}_x^{\alpha} = -k\left[\hat{\Theta}^{\alpha} \coth k\Delta - \frac{\hat{\Theta}^{\beta}}{\sinh k\Delta}\right]$$

$$\hat{v}_x^{\beta} = -k\left[\frac{\hat{\Theta}^{\alpha}}{\sinh k\Delta} - \hat{\Theta}^{\beta} \coth k\Delta\right]$$

Take $\mathcal{E}' = 0$

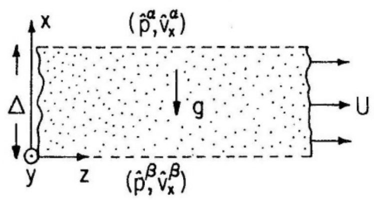
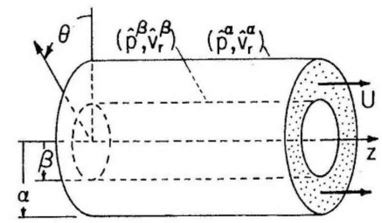
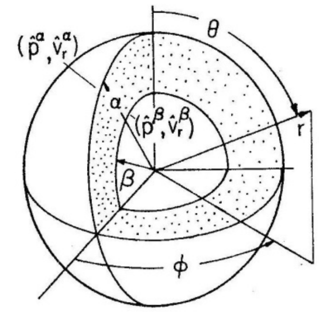
$$-\frac{d\hat{p}}{dx} = \rho j(\omega - k_z U)\hat{v}_x = -\rho j(\omega - k_z U)\frac{d\hat{\Theta}}{dx}$$

$$\hat{p} = \rho j(\omega - k_z U)\hat{\Theta}$$

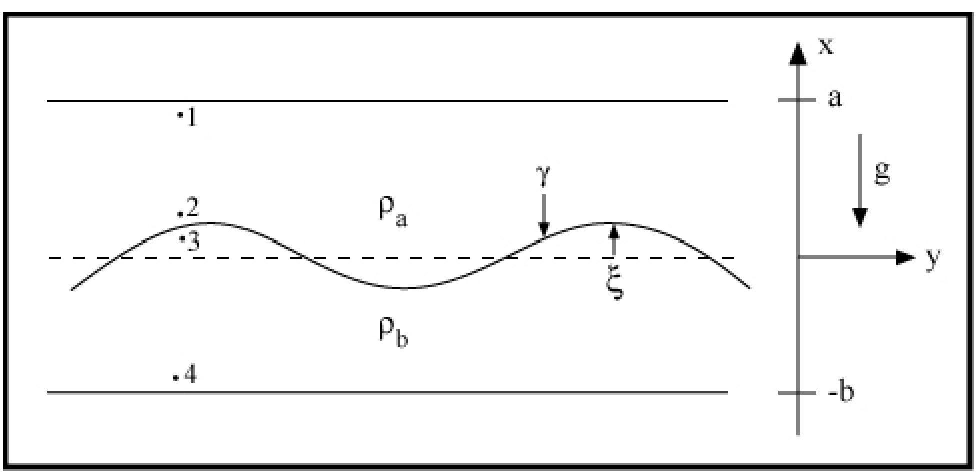
$$\begin{bmatrix} \hat{\Theta}^{\alpha} \\ \hat{\Theta}^{\beta} \end{bmatrix} = \frac{1}{k}\begin{bmatrix} -\coth k\Delta & \dfrac{1}{\sinh k\Delta} \\ -\dfrac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix}\begin{bmatrix} \hat{v}_x^{\alpha} \\ \hat{v}_x^{\beta} \end{bmatrix}$$

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = \frac{\rho j(\omega - k_z U)}{k}\begin{bmatrix} -\coth k\Delta & \dfrac{1}{\sinh k\Delta} \\ -\dfrac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix}\begin{bmatrix} \hat{v}_x^{\alpha} \\ \hat{v}_x^{\beta} \end{bmatrix}$$

Pressure-velocity relations for perturbations of inviscid fluid.

Cartesian	Cylindrical	Spherical
		
$p = \Pi - \frac{1}{2}\rho U^2 - \mathcal{E} - \rho g x + \text{Re}\,\hat{p}(x)e^{j(\omega t - k_y y - k_z z)}$ (a)	$p = \Pi - \frac{1}{2}\rho U^2 - \mathcal{E} + \text{Re}\,\hat{p}(r)e^{j(\omega t - m\theta - kz)}$ (d)	$p = \Pi - \mathcal{E} + \text{Re}\,\hat{p}(r)P_n^m(\cos\theta)e^{j(\omega t - m\phi)}$ (g)
$\hat{p}(x) = j(\omega - k_z U)\rho\hat{\Phi}(r)$ (b)	$\hat{p}(r) = j(\omega - kU)\rho\hat{\Phi}(r)$ (e)	$\hat{p}(r) = j\omega\rho\hat{\Phi}(r)$ (h)
$\begin{bmatrix}\hat{p}^\alpha \\ \hat{p}^\beta\end{bmatrix} = \frac{j(\omega - k_z U)\rho}{\gamma}\begin{bmatrix}-\coth\gamma\Delta & \frac{1}{\sinh\gamma\Delta} \\ \frac{-1}{\sinh\gamma\Delta} & \coth\gamma\Delta\end{bmatrix}\begin{bmatrix}\hat{v}_x^\alpha \\ \hat{v}_x^\beta\end{bmatrix}$ (c) $\gamma \equiv \sqrt{k_y^2 + k_z^2}$	$\begin{bmatrix}\hat{p}^\alpha \\ \hat{p}^\beta\end{bmatrix} = j(\omega - kU)\rho\begin{bmatrix}F_m(\beta,\alpha) & G_m(\alpha,\beta) \\ G_m(\beta,\alpha) & F_m(\alpha,\beta)\end{bmatrix}\begin{bmatrix}\hat{v}_r^\alpha \\ \hat{v}_r^\beta\end{bmatrix}$ (f) (See Table 2.16.2 for F_m and G_m)	$\begin{bmatrix}\hat{p}^\alpha \\ \hat{p}^\beta\end{bmatrix} = j\omega\rho\begin{bmatrix}F_n(\beta,\alpha) & G_n(\alpha,\beta) \\ G_n(\beta,\alpha) & F_n(\alpha,\beta)\end{bmatrix}\begin{bmatrix}\hat{v}_r^\alpha \\ \hat{v}_r^\beta\end{bmatrix}$ (i) (See Table 2.16.3 for F_n and G_n)
Compressible: $\gamma \equiv \sqrt{k_y^2 + k_z^2 - \frac{(\omega - k_z U)^2}{a^2}}$	Compressible; replace $k \rightarrow \gamma$ in F_m and G_m: $\gamma \equiv \sqrt{k^2 - \frac{(\omega - kU)^2}{a^2}}$	

II. Gravity – Capillary Dynamics



Equilibrium:

$$P_0 = \begin{cases} -\rho_a g x + P_0 & x > 0 \\ -\rho_b g x + P_0 & x < 0 \end{cases} \quad ; \ \bar{v} = 0 \ , \ \xi = 0$$

Perturbations:

$$\begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \end{bmatrix} = \frac{j\omega\rho_a}{k} \begin{bmatrix} -\coth ka & \dfrac{1}{\sinh ka} \\ -\dfrac{1}{\sinh ka} & \coth ka \end{bmatrix} \begin{bmatrix} \hat{v}_{x1} \\ \hat{v}_{x2} \end{bmatrix}$$

$$\begin{bmatrix} \hat{p}_3 \\ \hat{p}_4 \end{bmatrix} = \frac{j\omega\rho_b}{k} \begin{bmatrix} -\coth kb & \dfrac{1}{\sinh kb} \\ -\dfrac{1}{\sinh kb} & \coth kb \end{bmatrix} \begin{bmatrix} \hat{v}_{x3} \\ \hat{v}_{x4} \end{bmatrix}$$

$\hat{v}_{x1} = \hat{v}_{x4} \equiv 0$ (rigid boundaries)

Interface: $v_x = \dfrac{\partial\xi}{\partial t} + v_y \dfrac{\partial\xi}{\partial y} + v_z \dfrac{\partial\xi}{\partial z} \Rightarrow \hat{v}_{x2} = \hat{v}_{x3} = j\omega\hat{\xi}$

Force Balance

$$P_3'(\xi) - P_2'(\xi) = -\gamma\left(\frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right)$$

$$P_3(\xi) = P_{30}(0+\xi) + P_3'(\xi) = P_{30}(0) + \left.\frac{dP_{30}}{dx}\right|_{x=0} \xi + P_3'(0)$$

$$P_3'(\xi) = -\rho_b g\xi + P_3'(0)$$

$$P_2'(\xi) = -\rho_a g\xi + P_2'(0)$$

$$-\rho_b g\hat{\xi} + \hat{P}_3 + \rho_a g\hat{\xi} - \hat{P}_2 = \gamma k^2 \hat{\xi}$$

$$\hat{P}_3 = \frac{j\omega\rho_b}{k}\left[-\coth kb\,\hat{v}_{x3}\right] = +\frac{\rho_b\omega^2}{k}\coth kb\,\hat{\xi}$$

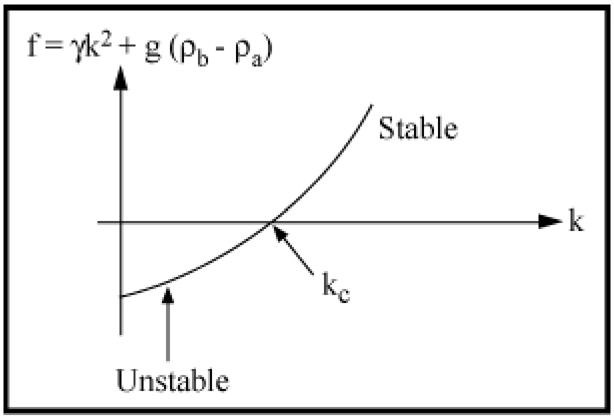
$$\hat{P}_2 = \frac{j\omega\rho_a}{k}\left[\coth ka\,\hat{v}_{x2}\right] = -\frac{\rho_a\omega^2}{k}\coth ka\,\hat{\xi}$$

$$\left[\frac{\omega^2}{k}\left(\rho_a \coth ka + \rho_b \coth kb\right) + g\left(\rho_a - \rho_b\right) - \gamma k^2\right]\hat{\xi} = 0$$

Dispersion Relation: $$\frac{\omega^2}{k}\left(\rho_a \coth ka + \rho_b \coth kb\right) = \gamma k^2 + g\left(\rho_b - \rho_a\right)$$

Instability if: $\gamma k^2 + g\left(\rho_b - \rho_a\right) < 0$ Rayleigh-Taylor Instability
(heavier fluid above)

if $\rho_a > \rho_b$



$$k_c = \left[\frac{g\left(\rho_a - \rho_b\right)}{\gamma}\right]^{1/2} = \frac{2\pi}{\lambda_T}$$

$$\lambda_T = 2\pi\left[\frac{\gamma}{g\left(\rho_a - \rho_b\right)}\right]^{1/2}$$ (Taylor Wavelength)

$\lambda > \lambda_T$ Unstable
$\lambda < \lambda_T$ Stable

Stable if $\rho_b > \rho_a$

Long Wavelength Limit: $ka \ll 1,\ kb \ll 1,\ \gamma k^2 \ll g\left(\rho_b - \rho_a\right)$
$\coth ka \approx 1/ka$
$\coth kb \approx 1/kb$

$$\frac{\omega^2}{k^2}\left(\frac{\rho_a}{a} + \frac{\rho_b}{b}\right) = g\left(\rho_b - \rho_a\right)$$

$$\frac{\omega^2}{k^2} = v_P^2 = \frac{g\left(\rho_b - \rho_a\right)}{\frac{\rho_a}{a} + \frac{\rho_b}{b}}$$ Non-dispersive gravity wave