MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

Receivers, Antennas, and Signals – 6.661

Solutions -- Problem Set No. 2 December 22, 2003

Problem 2.1

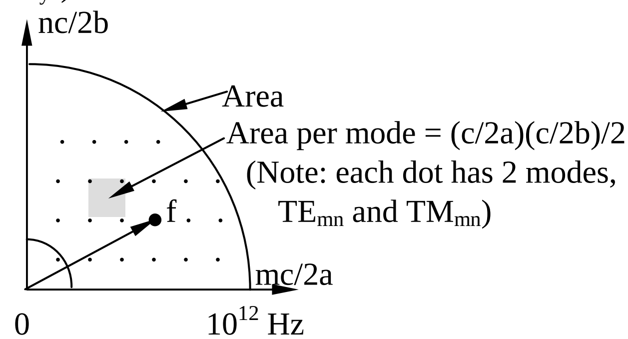
a) $T_{up} = 3e^{-\tau} + 250(1 - e^{-\tau}) = 3\times0.6 + 250\times0.4 = \boxed{101.8K}$

b) $T_{30} = 3e^{-2\tau} + 250(1 - e^{-2\tau}) = 3\times(0.6)^2 + 250(1 - 0.6^2) = \boxed{161.08K}$

Problem 2.2

a) To count the propagating modes, consider the mode diagram, where the axes represent f_x and f_y and where $f_{cutoff} = (f_x^2 + f_y^2)^{0.5}$:

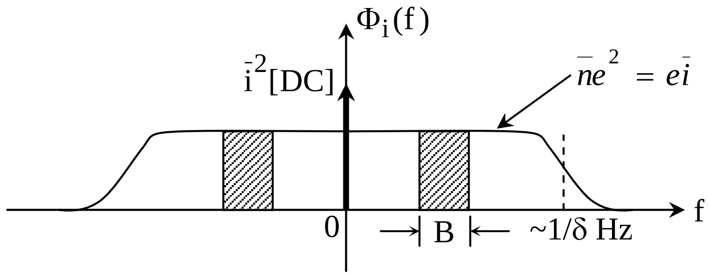
The number N of propagating modes is the total area divided by the area for each mode. Thus $N = 2(\pi r^2/4)/(c^2/4ab)$
$N = 2\pi ab10^{24}/c^2$
$= 2\times3.14\times2\times10^{-2}\times10^{-2}10^{24}/(3\times10^8)^2$
$= \boxed{13{,}956 \text{ modes}}$ (approximately).



b) Referring to the figure, each mode propagates from its own cutoff frequency ("f" in the figure) up to 10^{12} Hz; thus $B = (10^{12} - f)$. The total thermal power radiated for mode i is then the integral from f to 10^{12}, or kTB_i for each mode, where the number density of modes/Hz at frequency f is $2(f\pi/2)/(c/2b)(c/2a)$. The factor of 2 corresponds to the fact that mn corresponds to both a TE and a TM mode. Therefore,
$P = 2kT\int_0^{10^{\wedge}12} (f\pi/2)(10^{12} - f)df/(c/2b)(c/2a)$. Therefore,
$P = (4abkT\pi/c^2)\int_0^{10^{\wedge}12} f(10^{12} - f)\,df = (4\pi abkT/c^2)[10^{12} f^2/2 - f^3/3]|_0^{10^{\wedge}12}$
$= (4\pi abkT/c^2)10^{36}/6 = \boxed{1.9 \times 10^{-5}\{\text{watts}\}}$

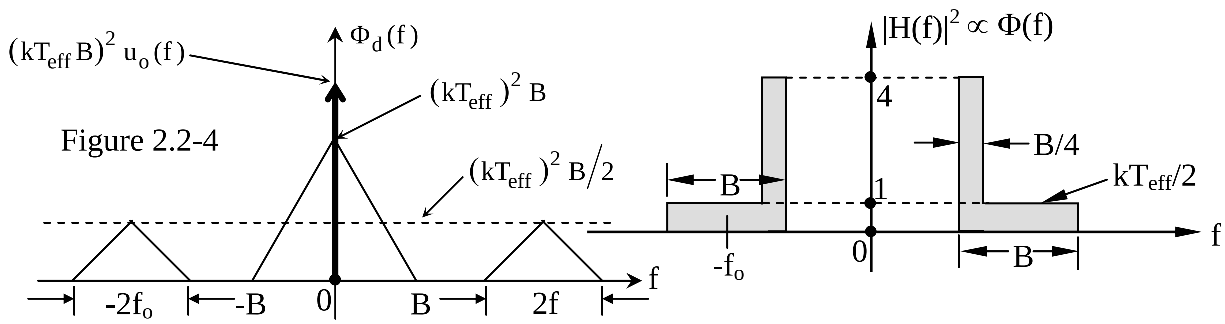
Problem 2.3

The figure for $\Phi_i(f)$ shows the two elements that we need to derive, i.e., the DC term $= \bar{i}^2$, and the AC term with a low-frequency value of $\bar{i}e$. The DC term of $\Phi(f)$ is simply the magnitude squared of the DC term of the fourier transform of the impulse train. The DC value of the impulse train i(t) is $\bar{n}e = \bar{i}$, and its square is $\bar{i}^2$.



The AC terms follows from $\Phi_i(f) \leftrightarrow \phi_i(\tau) = E[i(t)i(t - \tau)]$ The current i(t) is a poisson-distributed train of brief current pulses each with time integral = e, where e is the charge on an electron. The expectation E has two parts, one for pulses being multiplied by themselves, and one for the product of independent pulse pairs; the second part corresponds to DC and has already been evaluated. The non-DC part of $\phi(\tau)$ is $\bar{n}\phi'(\tau)$, where $\phi'(\tau)$ corresponds to a single pulse. But the fourier transform $\underline{I}(f)$ of a single pulse i(t) of area e is a ~DC signal of value e that extends to an upper frequency limit ~$1/\delta$, where δ is the nominal duration [seconds] of the pulse. The corresponding energy density spectrum for such a single pulse has a low-frequency value of e^2 that extends to $1/\delta$ Hz, and its fourier transform is $\phi'(\tau)$. Therefore the low-frequency value of $\Phi_i(f)$ is the fourier transform of $\bar{n}\phi'(\tau)$, or $\bar{n}e^2 = \bar{i}e$. Q.E.D.

Problem 2.4



The derivation in the text and Figure 2.2-4 must be altered; only the new values of $\Phi_d(f)$ near f = 0 are of interest, where:

$$\phi_d(\tau) = \overline{v_i^2(t)}\,\overline{v_i^2(t-\tau)} + 2\,\overline{v_i(t)v_i(t-\tau)}^2 = \phi_i^2(0) + 2\phi_i^2(\tau)$$

Since: $\phi_i(0) = \overline{v_i^2(t)} = \int_{-\infty}^{\infty} \Phi_i(f)\,df = kT_{eff}B(1 + 0.75)$ (see figure above),

the impulse at the origin of $\Phi_d(f)$ becomes $(1.75\ kT_{eff}B)^2 u_o(f)$ while the AC part near zero frequency has value $2\Phi(f)*\Phi(f)$ for $f \cong 0$. Referring to the figure above for $\Phi(f)$, we find that the AC part near f = 0 is $2([2B/4][2kT_{eff}]^2 + [6B/4][kT_{eff}/2]^2) = [19B/4][kT_{eff}]^2$.

The DC power P_{DC} emerging from the output filter is:

$$\Phi_{o_{DC}}(f) = \left(1.75kT_{eff}B\right)^2 (A\tau)^2 u_o(f) \qquad (2.2.10)$$

The variance P_{AC} of the fluctuating component of the output voltage is $\Phi_{dAC}(0)$:

$$P_{AC} = \int_{-\infty}^{\infty} \Phi_{oAC}(f)df \cong [19B/4][kT_{eff}]^2 \int_{-\infty}^{\infty} |H(f)|^2 df = [19B/4][kT_{eff}]^2 A^2\tau$$

Therefore Equation (2.2.13) becomes:

$\Delta T_{rms} = P_{AC}^{0.5}/(\partial P_{DC}^{0.5}/\partial T_A) = [19B\tau/4]^{0.5}[kT_{eff}]A\tau^{0.5}/(1.75kBA\tau) = \boxed{1.25T_{eff}/(B\tau)^{0.5}\ K}$

Problem 2.5

ΔT_{rms} = (variance of integrator output v_o)$^{0.5}$/($M\langle v_o\rangle/MT_A$)
$\sigma_o^2 = 2B\tau\sigma_d^2$ = variance of integrator output v_o
$\sigma_d^2 = \langle(v_i^2 - \langle v_i^4\rangle)^2\rangle = \langle v_i^8\rangle - 2\langle v_i^4\rangle^2 + \langle v_i^4\rangle^2 = \langle v_i^8\rangle - \langle v_i^4\rangle^2$
Let $\langle v_i^2\rangle = aT_{eff}\langle x^2\rangle$ where $\langle x^2\rangle = 1$ and $\langle x^n\rangle = (n-1)(n-3)\ldots 1$ for n even; then
$\sigma_d^2 = (aT_{eff})^4 (\langle v_i^8\rangle - \langle v_i^4\rangle^2) = (aT_{eff})^4(105 - 9) = 96(aT_{eff})^4$
$\langle v_o\rangle = 2B\tau\langle v_i^4\rangle = 2B\tau T_{eff}^2 a^2\langle x^4\rangle$ and $T_{eff} = T_A + T_R$, $T_{eff}^2 = T_A^2 + 2T_AT_R + T_R^2$
$M\langle v_o\rangle/MT_A = 6B\tau a^2(2T_A + 2T_R) = 12B\tau\ T_{eff}\ a^2$, so
$\Delta T_{rms} = \sigma_o/(M\langle v_o\rangle/MT_A) = 2B\tau 96(aT_{eff})^4/(12B\tau\ T_{eff}\ a^2) = \boxed{(4/3)^{0.5}T_{eff}/(B\tau)^{0.5}\ \text{Q.E.D.}}$