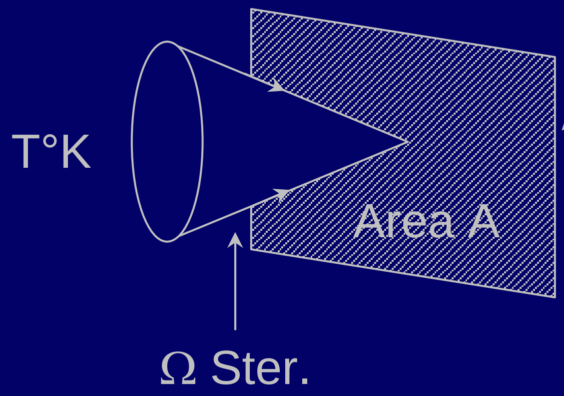
Photon noise



$$\sigma_p(\text{watts}) = \sqrt{n_s \sigma_n^2 \, hf/\tau}$$

$\left(\begin{array}{l}\text{number of states}\\ \text{in cavity in } B\tau\end{array}\right)$

Calculation of the number of electromagnetic modes = m:

Recall, for hf << kT, $kT \ \text{W Hz}^{-1} \bullet \text{mode}^{-1} \Rightarrow \dfrac{2kT}{\lambda^2} \text{W Hz}^{-1} m^{-2} \text{ster}^{-1}$

Therefore $\dfrac{2kT}{\lambda^2} \Big/ kT = \dfrac{2}{\lambda^2} = 2(f/c)^2 \left(\dfrac{\text{modes}}{m^2 \bullet \text{ster}}\right) \neq f(T)$

Therefore $\boxed{m = 2(f/c)^2 A\Omega \ \text{propagation modes}}$

Photon noise

Therefore $\boxed{m = 2(f/c)^2 A\Omega \text{ modes}}$ $\sigma_p(\text{watts}) = \sqrt{n_s \sigma_n^2}\, hf/\tau$

Each mode has $2B\tau$ degrees of freedom
(2B samples sec^{-1} times τ sec) (Nyquist sampling)

Each energy state (ε = hf) has 2 degrees of freedom
(sin ωt, cos ωt)

Therefore number of states n_s @ hf in $B\tau$:

$$n_s = (\#\text{ modes in } A\Omega) \bullet \left(\frac{\text{degrees}}{\text{mode}}\right) \bullet \left(\frac{\text{states}}{\text{degree}}\right)$$

$$= (m)(2B\tau)(1/2) = 2(f/c)^2 A\Omega(2B\tau)(1/2)$$

Photon noise

$$\sigma_p = \sqrt{n_s \sigma_n^2}\, hf/\tau \qquad n_s = 2(f/c)^2 A\Omega B\tau \qquad \sigma_n^2 = \bar{n} + \bar{n}^2$$

$$\overline{\text{photons/state}} = \bar{n} = \frac{1}{e^{hf/kT} - 1}$$

Therefore $\sigma_p = \sqrt{2\left(\frac{f}{c}\right)^2 A\Omega B\tau\left[\bar{n} + \bar{n}^2\right](hf)^2 \Big/ \tau^2}$ W "quantum limit"

If boxcar h(t) has τ = 0.5 sec, $\equiv$ 1-Hz post-detection bandwidth,

yields units of $\mathrm{W\,Hz}^{-1/2}$ (NEP_R)

$$NEP_R(f) = \sqrt{4A\Omega\left(\frac{f}{c}\right)^2 B\left[\bar{n} + \bar{n}^2\right](hf)^2}\ \mathrm{W\,Hz}^{-1/2}$$

"Noise-equivalent power" due to radiation noise

Photon noise

$$NEP_R(f) = \sqrt{4A\Omega\left(\frac{f}{c}\right)^2 B\left(\bar{n}+\bar{n}^2\right)(hf)^2} \quad W\,Hz^{-1/2}$$

"Noise-equivalent power" due to radiation noise

NEP_R for a blackbody, all frequencies $(B \rightarrow \infty)$

$$NEP_{R\infty} = \left[4A\Omega\int_0^\infty \left(hf^2/c\right)^2\left(\bar{n}+\bar{n}^2\right)df\right]^{1/2} \quad \text{where} \quad \bar{n}(f) = \frac{1}{e^{hf/kT}-1}$$

$$NEP_{R\infty} = \left[4A\Omega(4kT)\frac{\sigma_{SB}}{\pi}T^4\right]^{1/2}\left[WHz^{-1/2}\right]$$

Where $\sigma_{SB} \equiv$ "Stefan-Boltzmann constant" = $5.67 \times 10^{-8}\ Wm^{-2}K^{-4}$
and Hz^{-1} refers to the detector output bandwidth

Photon noise

$$NEP_{R\infty} = \left[4A\Omega(4kT)\frac{\sigma_{SB}}{\pi}T^4\right]^{1/2} WHz^{-1/2}$$

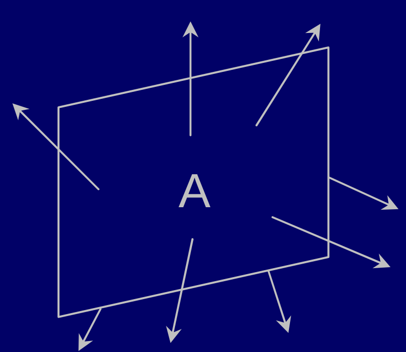
Where $\sigma_{SB} \equiv$ “Stefan-Boltzmann constant” = $5.67 \times 10^{-8}\ Wm^{-2}K^{-4}$

Recall: blackbodies radiate

$$P_r = A\Omega\frac{\sigma_{SB}}{\pi}T^4 \text{ watts } (\text{small } \Omega), P_r = A\sigma_{SB}T^4 \text{ if } \Omega = 2\pi$$

Therefore: $\boxed{NEP_{R\infty} = [P_r 16kT]^{1/2} WHz^{-1/2}}$ for $\Omega = 2\pi$

Therefore minimize T,A



$\Omega = 4\pi$

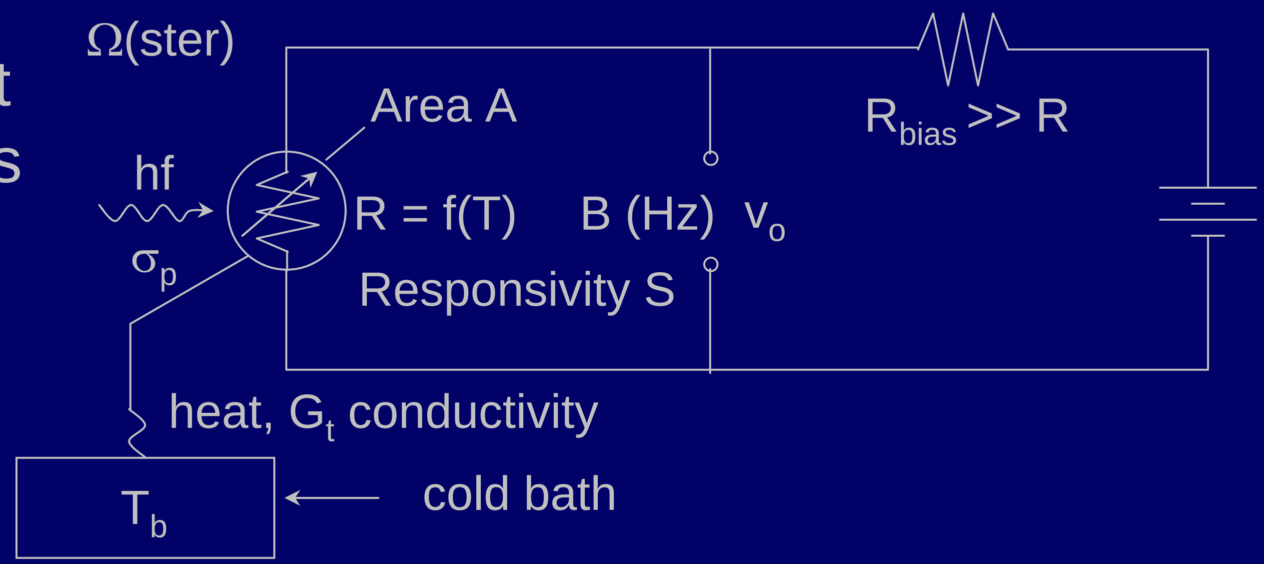
$$NEP_{R\infty} = 4\sqrt{A\sigma_{SB}kT^5}$$

$$\cong 4\times10^{-16} WHz^{-1/2}$$

$$\text{for } A = 10^{-6}, T = 4°K$$

Bolometer noise analysis

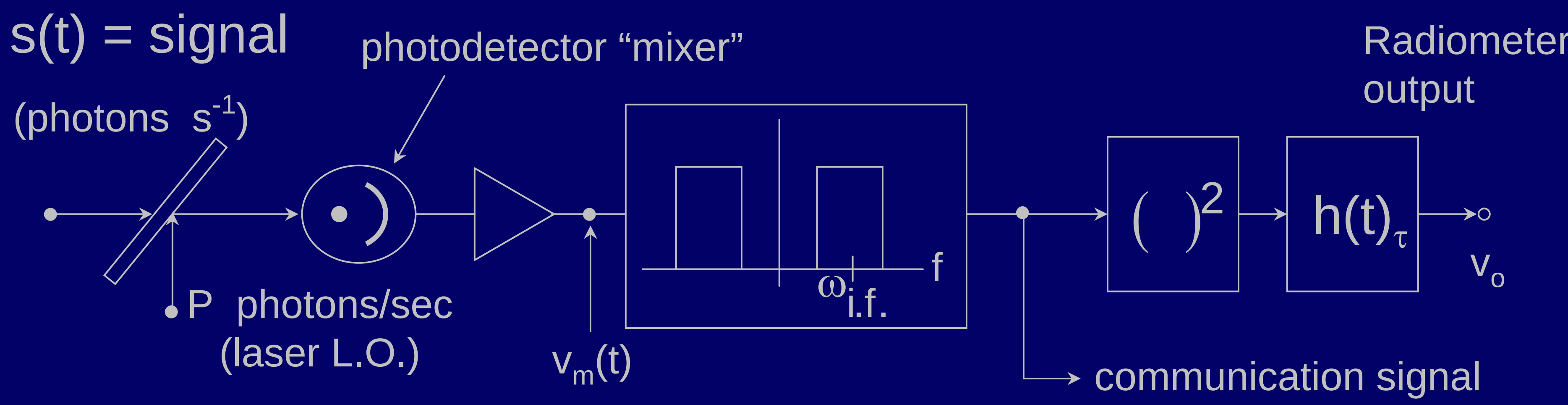
Bolometers first convert photons to heat



R, R_b produce Johnson noise
Radiated photons have shot noise, i.e. "radiation noise"
"Phonon noise" arises from shot noise in phonons carrying heat to the cold bath

$$NEP = \sqrt{\underbrace{4kT_bR/S^2}_{\text{Johnson noise}} + \underbrace{16A\Omega kT^5\sigma_{SB}/\pi}_{\text{Photon shot noise}} + \underbrace{4kG_tT_b^2}_{\text{Phonon shot noise}}}\ \left(W\ Hz^{-1/2}\right)$$

Optical Superheterodynes



Let signal $s(t) = \sqrt{2S}\cos\omega_s t \propto \bar{E}(t)$, $\overline{s^2(t)} \triangleq S$

local oscillator $p(t) = \sqrt{2P}\cos\omega_o t$ Both are coherent lasers

D = dark photons/sec

Mixer output (L.O. $\equiv$ 0) = $\eta S + D$ counts/sec, where $\eta \equiv$ quantum efficiency

$$v_m(t) = \text{constant} \bullet \eta\left(\sqrt{2S_o}\cos\omega_s t\right)^2 \quad \text{(volts)}$$

$$= \text{constant}\left[\eta\left(S + P + \sqrt{SP}\cos\omega_{i.f.}t\right) + D\right] \text{where } \omega_{i.f.} = \left|\omega_s - \omega_o\right|$$

We want $P >> S$, $\eta\sqrt{SP} >> D$

Optical Superheterodyne CNR

$$v_m(t) = \text{constant} \bullet \eta\left(\sqrt{2S_o}\cos\omega_s t\right)^2 \quad \text{(volts)}$$

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$$\text{We want } P >> S,\ \eta\sqrt{SP} >> D$$

Let "constant" $\triangleq$ 1 so $v_m(t)$ units are counts/sec

$$v_{mix}(t) \cong v_m[\text{signal}] + v_m[\text{noise}]$$

$v_m[\text{signal}] \cong \eta\sqrt{SP}\cos\omega_{i.f.}t$ Conveys information in $S(t), \omega_{i.f.}$

$$v_m[\text{rms noise}] \cong \sqrt{2\eta P}\left(\frac{\text{counts/sec}}{\sqrt{\text{Hz}}}\right)\left[= \sqrt{\frac{\eta P}{\tau}},\ \tau = 0.5 \text{ sec for } \sqrt{\text{Hz}}\right]$$

$$\left(\cong \sqrt{2D} \text{ if } D >> \eta P\right)$$

$\bar{v}_o = \eta^2 SP\left\langle\cos^2\omega_{i.f.}t\right\rangle$ Conveys information in s(t) for $\omega \lesssim \dfrac{2\pi}{\tau} << \omega_{i.f.}$

Optical Superheterodyne CNR

Let "constant" $\triangleq$ 1 so $v_m(t)$ units are counts/sec

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$\bar{v}_o = \eta^2 SP\langle\cos^2\omega_{i.f.}t\rangle$ Conveys information in s(t) for $\omega \lesssim \frac{2\pi}{\tau} << \omega_{i.f.}$

$$v_{o_{\text{rms noise}}} = \left(\sqrt{2\eta PB}\right)^2 \Big/ \sqrt{B\tau} \text{ for } P >> S,\ P >> D$$

Define CNR "Carrier-to-Noise Ratio" for $v_m(t) = \eta^2 SP\Big/\left[4\eta PB/\sqrt{B\tau}\right] = \eta S\sqrt{\tau/16B}$

We assume τ > 1/B so τ provides additional noise smoothing

$$\boxed{\text{CNR} \gtrsim \eta S\tau/4}$$

(best we could do is CNR $\leq$ 1 for 4 photons/bit)

Optical Superheterodynes, Comparisons

1) Radio total-power radiometer:

$$\text{CNR} = T_A / \Delta T_{RMS} = T_A \sqrt{B\tau}/T_R \quad \text{for} \quad T_R >> T_A$$

$$\text{Optical superheterodyne CNR} = \underbrace{\eta S}_{\text{photons sec}^{-1}} \sqrt{\tau/16B} = \overbrace{\underbrace{\eta(kT_A B/hf)}_{\text{photons sec}^{-1}} \sqrt{\tau/16B}}^{\text{radio expression}}$$

Therefore "T_R" = $4hf/k\eta$ if CNR (radio) = CNR (optical)

This is 4 times radio quantum limit if i.f. noise etc. is negligible

(Note: $P_A \neq kT_AB$ in optical)

Thus optical superheterodynes can approach quantum limit

Optical Superheterodynes, Comparisons

2) Optical non-superheterodyne if D >> S; then

$$v_{o_{sig}} = \eta S \text{ (gain normalized)}$$

$$v_{o_{noise}} = \sqrt{2DB} \Big/ \sqrt{B\tau}$$

$$CNR = \eta S \sqrt{\tau / 2D} \quad \text{versus} \quad CNR_{S.H.} = \eta S \sqrt{\tau / 16B}$$

Therefore a superheterodyne is better if $B_{i.f.} < D/8$

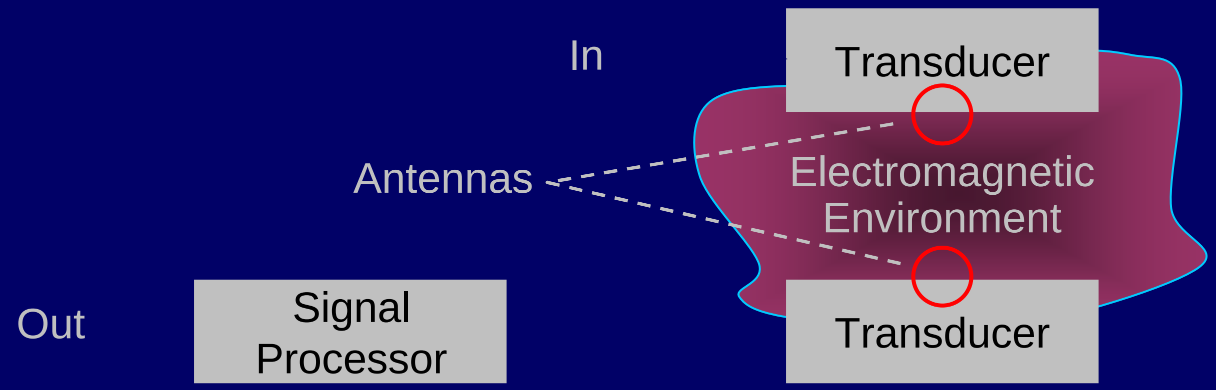
i.e. the worse D is, the higher B can be before L.O. shot noise dominates (assuming no mixer or i.f. excess noise)

Antennas

Basic Characterization

Professor David H. Staelin

Uses of Antennas

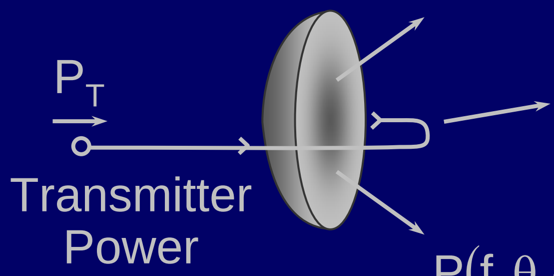


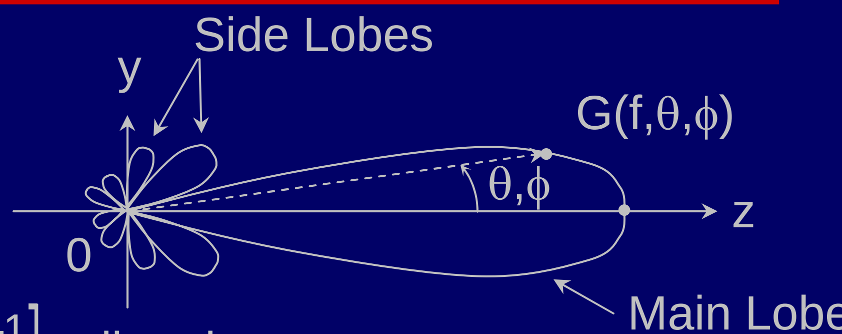
Antennas couple electromagnetic radiation and transmission lines for transmission and reception

We have studied: $hf \ll kT$ Radio
$hf \gg kT$ Optical
$hf \approx kT$ IR

All bands use antennas

Antennas – Characterization





$P(f,\theta,\phi)\ \left[\mathrm{W\ ster}^{-1}\right]$ radiated power

$\Downarrow$

$P_{TR}[\mathrm{W}] = \int_{4\pi} P\,d\Omega$ "total radiated power"

Radiation efficiency: $\eta_R \triangleq P_{TR}/P_T$

Gain (over isotropic): $G(f,\theta,\phi) \triangleq \dfrac{P(f,\Theta,\phi)}{P_T/4\pi} = \eta_R D(f,\theta,\phi)$

Directivity (over isotropic): $D(f,\theta,\phi) \triangleq \dfrac{P(f,\theta,\phi)}{P_{TR}/4\pi}$

Antenna pattern: $t(f,\theta,\phi) \triangleq \dfrac{G(f,\theta,\phi)}{G_o} \triangleq \dfrac{D(f,\theta,\phi)}{D_o} \le 1$

Antenna Example

$$P\left(Wm^{-2}\right) = G(\theta,\phi) \bullet \frac{P_T}{4\pi R^2}$$

Target	R(meters)	P(Wm^{-2})
Moon	3×10^{8}	10^{-5}
Jupiter	10^{12}	10^{-12}
Antares	3×10^{16}	10^{-21}

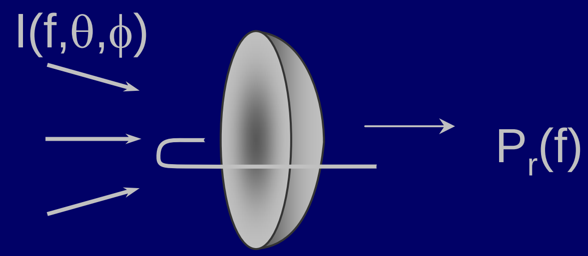
MIT "Haystack" antenna @ λ =1 CM; $G_o \cong$ 73 dB
Assume it radiates 1–MWatt radar pulses

Assume kTB $\cong 1.4 \times 10^{-23} \times 10°K \times 1Hz \cong 10^{-22}$ Watts(say $T_R \cong 10°$ and we use 1–Hz CW radar)

Then P(Wm^{-2}) received on Antares is comparable to receiver noise power kTB

Receiving Properties of Antennas

Characterized by Effective Area $A(f, \theta, \phi)$:



Power spectral density received:

$$P_r(f) = \underbrace{A(f,\theta,\phi)}_{[m^2]} \bullet \underbrace{[I(f,\theta,\phi) \bullet \Delta\Omega]}_{\text{"flux density"}\left[S\left(Wm^{-2}Hz^{-1}\right)\right]} \Delta f\,(W)$$

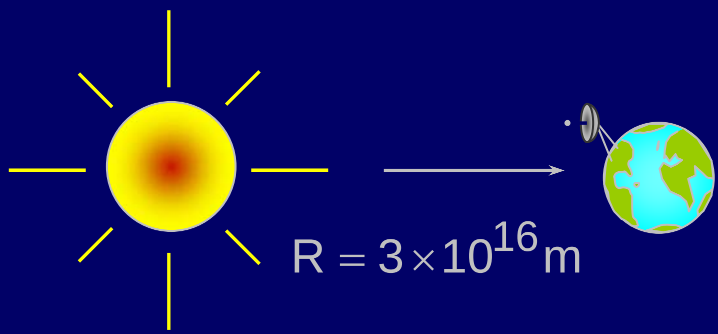
Δf, $\Delta\Omega$ are source bandwidth, solid angle

Recall: Radiation intensity $I(f,\theta,\phi)$ received from blackbody at temperature T is:

$$I(f,\theta,\phi) = \frac{2kT}{\lambda^2}\left[Wm^{-2}Hz^{-1}ster^{-1}\right]$$

Receiving System Example

Recall – 1-MW radar on Antares $\Rightarrow 10^{-21}$ W/m^2 on earth (G_T = 73 dB)

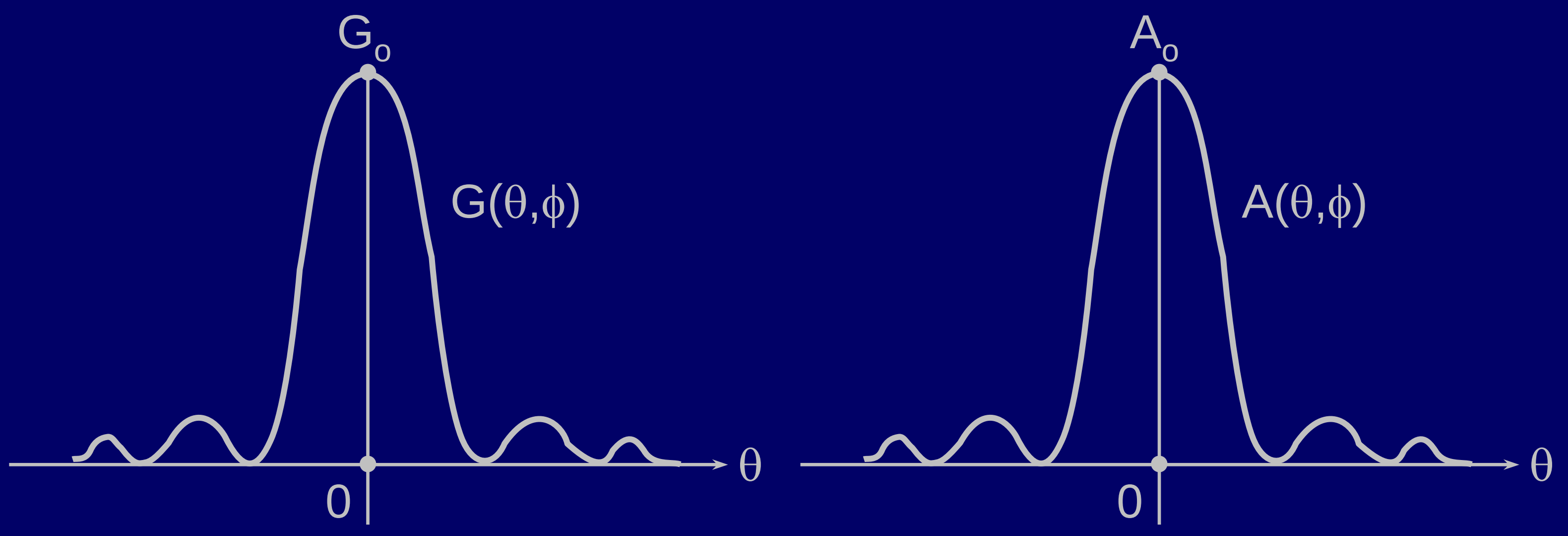


$$\text{Received power} = \underbrace{A(f,\theta,\phi)}_{\text{say } 10^4 \text{m}^2} \bullet \underbrace{\underbrace{\left[I(f,\theta,\phi)d\Omega\right]}_{S}}_{10^{-21}\text{Wm}^{-2}\text{Hz}^{-1}} \bullet B = 10^{-17}\text{ W from Antares}$$

Suppose kTB = 10^{-22}W (recall above) then SNR = 10^5

Audio at 10^4Hz $\Rightarrow$ SNR = 10 (commercial opportunity?)

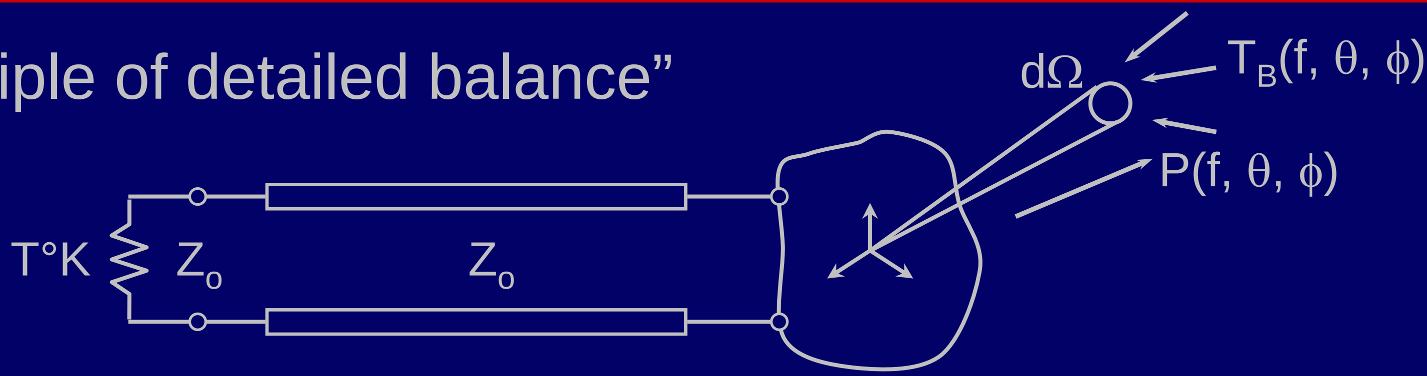
Relation Between A(f, θ,φ) and G(f, θ,φ)



We later prove (using reciprocity) that $\dfrac{G(\theta,\phi)}{G_o} = \dfrac{A(\theta,\phi)}{A_o}$

Receiving Properties Deduced from Reciprocity and Thermodynamics

"Principle of detailed balance"



Reciprocity + thermal equilibrium says that within dΩ, power out = power in

$$\text{Antenna radiates } P(f,\theta,\phi)\left[\text{W Hz}^{-1}\text{ster}^{-1}\right] \text{into } d\Omega\text{, so}$$

$$\text{power out} = P(f,\theta,\phi)\,d\Omega\,df = (kT\,df/4\pi)\,G\,d\Omega\,(\text{watts})$$

Antenna receives from dΩ:

$$\text{power in} = \underset{\text{One polarization}}{\frac{1}{2}}\underbrace{\frac{2kT_B(\theta,\phi)}{\lambda^2}}_{\text{Wm}^{-2}\text{Hz}^{-1}\text{ster}^{-1}}df\;d\Omega\;A(f,\theta,\phi)[\text{watts}]$$

Receiving Properties Deduced from Reciprocity and Thermodynamics

Antenna radiates $P(f, \theta,\phi)$ [W Hz^{-1} ster^{-1}] into $d\Omega$, so power out = $P(f, \theta,\phi)\, d\Omega\, df = (kTdf/4\pi)\, G d\Omega$ (watts)

Antenna receives from $d\Omega$:

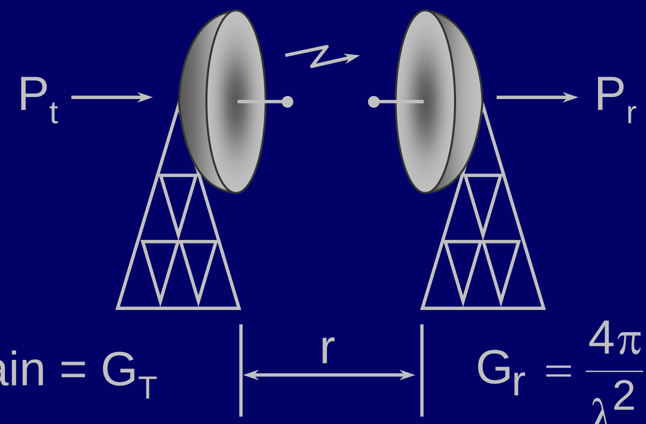
$$\text{power in} = \underset{\text{One polarization}}{\frac{1}{2}} \underbrace{\frac{2kT_B(\theta,\phi)}{\lambda^2}}_{\text{Wm}^{-2}\text{Hz}^{-1}\text{ster}^{-1}} df\, d\Omega\, A(f,\theta,\phi)[\text{watts}]$$

In thermal equilibrium $T = T_B(f, \theta, \phi)$; then equating radiation and reception (detailed balance) yields

$$\boxed{G(f,\theta,\phi) = \frac{4\pi}{\lambda^2} A(f,\theta,\phi)}$$

This assumes hf << kT and that powers superimpose, i.e., that the $T_B(\theta_1,\phi_1)$ signal $\bar{E}(t)$ is uncorrelated with that for $T_B(\theta_2,\phi_2)$

Antennas Used to Provide a Radio Link



Gain = G_T $\qquad G_r = \frac{4\pi}{\lambda^2} A_r$

$$P_r = \overbrace{\underbrace{\frac{P_t}{4\pi r^2}}_{\text{isotropic}} \bullet G_t}^{\text{Wm}^{-2}\text{ at receiver}} \bullet \underbrace{A_r}_{\text{m}^2\text{ effective area}} \text{ Watts}$$

Note: $P_r \rightarrow \infty$ as $r \rightarrow 0$!, so this relation requires r > r minimum

Let $P_r = P_t$ at r_{min} and $A_t = A_r = D^2$ (m^2) [D ≅ aperture diameter in practice]

Then $\frac{G_t A_r}{4\pi r_{min}^2} = 1 = \frac{A_t A_r}{\lambda^2 r_{min}^2} = \frac{D^4}{\lambda^2 r_{min}^2}$

Therefore $r_{min} = D^2/\lambda$ (in practice we want $r > 2D^2/\lambda$)

This zone where $r \gtrsim 2D^2/\lambda$ is called the "far field" of the aperture

Definition of Antenna Temperature T_A(°K)

$$kT_A \left(W\ Hz^{-1}\right) = \int_{4\pi} \underbrace{\overbrace{A(\theta,\phi)}^{\left(m^2\right)} \overbrace{I(\theta,\phi)}^{\left(Wm^{-2}Hz^{-1}ster^{-1}\right)}}_{\text{for a specific polarization}} d\Omega$$

Received power spectral density

Since $I = \frac{2kT_B}{\lambda^2} \bullet \frac{1}{2}$ for thermal radiation, single polarization

Therefore

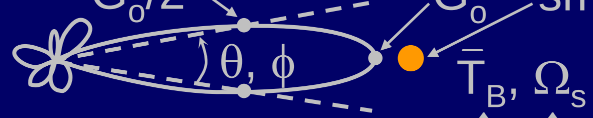
$$\boxed{\begin{aligned} T_A &= \frac{1}{\lambda^2} \int A(\theta,\phi) T_B(\theta,\phi)\, d\Omega \\ &= \frac{1}{4\pi} \int G(\theta,\phi) T_B(\theta,\phi)\, d\Omega \end{aligned}}$$

For T_B uncorrelated in angle

Observing Small Thermal Sources $T_B(\theta,\phi,f)$

Limiting case:

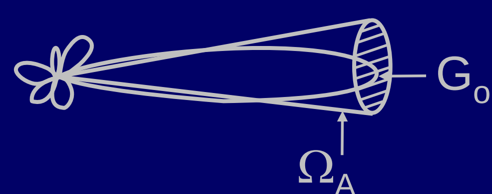
$$T_{A_S} = \frac{1}{4\pi}\int_{\Omega_S} G(\theta,\phi)\, T_B(\theta,\phi)\, d\Omega = \frac{G_o \overline{T}_B \Omega_S}{4\pi} = \frac{A_o \overline{T}_B \Omega_S}{\lambda^2}$$



T_{A_S} is due to source (assume zero background)

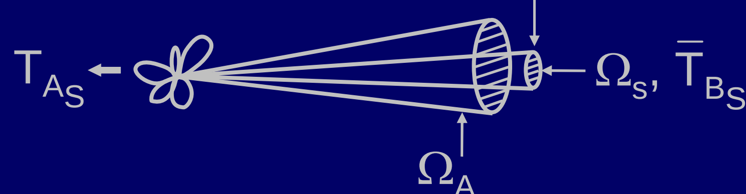
Physical interpretation (for $\theta_S << \theta_B$):

Define Ω_A, "beam solid angle" so that $G_o \Omega_A \triangleq \int_{4\pi} G\, d\Omega = 4\pi\eta_r$



$$G_o = 4\pi\eta_r / \Omega_A$$

Then $T_{A_S} \triangleq \underbrace{\left(\frac{\Omega_S}{\Omega_A}\eta_r\right)}_{\text{Coupling coefficient} \leq \eta_r} \overline{T}_B$



Geometric coupling ratio $= \dfrac{\Omega_S}{\Omega_A}$

Ways to Characterize Small Thermal Sources

Limiting case:

$$T_{A_S} = \frac{1}{4\pi}\int_{\Omega_S} G(\theta,\phi)\, T_B(\theta,\phi)\, d\Omega = \frac{G_o T_B \Omega_S}{4\pi} = \frac{A_o T_B \Omega_S}{\lambda^2}$$

1. $T_B(\theta,\phi,f)$ (for each of 2 polarizations)

2. $\overline{T}_{B_S}$ average brightness temperature

3. $S(f)\left[Wm^{-2}Hz^{-1}\right] = \int_{\Omega_S} I(f,\theta,\phi)\, d\Omega \neq f(\text{antennas})$

   if source small, $\Omega_S << \Omega_A$

Units of S: 1 "jansky" $= 10^{-26}$ $Wm^{-2}Hz^{-1}$