Lecture 18: Ginzburg-Landau Theory

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Ginzburg-Landau Expansion

Ginzburg-Landau Theory focuses on the Free Energy Difference between two phases

It assumes that one phase is related to another by a small parameter which changes continuously near the transition from one phase to another.

Let's recall the free energy difference between the superconducting and normal phases

$$G_s(0,T) - G_n(0,T) \equiv -\frac{1}{2}\,\mu_o H_c^2(T)\,V_s$$

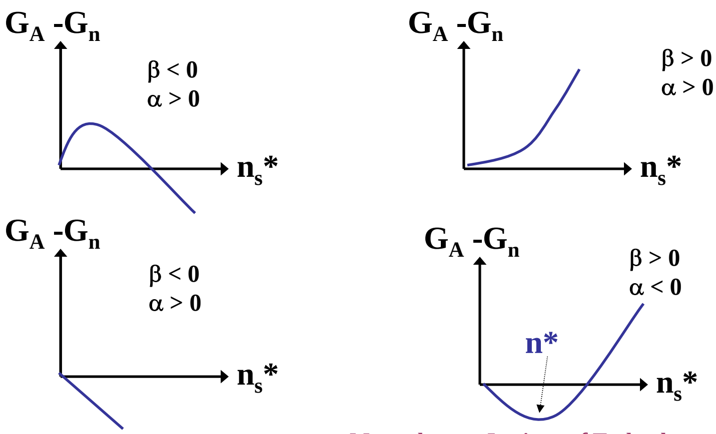
Assume the free energy of the phase A evolves from the normal state at no field as a power series in the density of superconducting electrons:

$$G_A(0,T,n_s^\star) = G_n(0,T) + \int_{V_s} \left(\alpha(T) n_s^\star + \frac{1}{2}\beta(T) {n_s^\star}^2 + \dots\right) dv$$

Homogenous Material

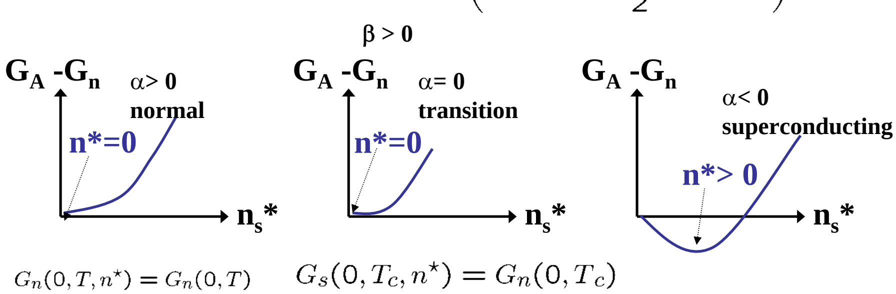
Consider the case where the density is uniform, then

$$G_A(0,T,n_s^\star) - G_n(0,T) = V_s\left(\alpha(T)n_s^\star + \frac{1}{2}\beta(T)n_s^{\star\,2}\right)$$



Homogenous Material

$$G_A(0,T,n_s^\star) - G_n(0,T) = V_s\left(\alpha(T)n_s^\star + \frac{1}{2}\beta(T)n_s^{\star\,2}\right)$$



$G_n(0,T,n^\star) = G_n(0,T)$ $\quad$ $G_s(0,T_c,n^\star) = G_n(0,T_c)$

For the superconducting state with **T <T$_C$**

$$\left(\frac{\partial}{\partial n_s^\star} G_A(0,T,n_s^\star)\right)\Bigg|_{n_s^\star = n^\star} = 0 \quad \Rightarrow \quad n^\star = -\frac{\alpha}{\beta}$$

α and β in terms of physical paramaters

In Summary: $$G_A(0,T,n_s^\star) - G_n(0,T) = V_s\left(\alpha(T)n_s^\star + \frac{1}{2}\beta(T){n_s^\star}^2\right)$$

$$n^\star = \begin{cases} 0 & \text{for } \alpha > 0 \\ -\dfrac{\alpha}{\beta} & \text{for } \alpha < 0 \end{cases} \qquad G_A(0,T,n^\star) = \begin{cases} G_n(0,T) & \text{for } \alpha > 0 \\ G_n(0,T) - \underbrace{\dfrac{1}{2}\dfrac{\alpha^2}{\beta}}_{\mu_o H_c^2(T)} V_s & \text{for } \alpha < 0 \end{cases}$$

Therefore, $\alpha = -\dfrac{\mu_o H_c^2}{n^\star}$ and $\beta = \dfrac{\mu_o H_c^2}{(n^\star)^2}$

So writing the free energy in terms of H_C and n*

$$G_s(0,T,n_s^\star) = G_n(0,T) - \mu_o H_c^2 V_s\left(\frac{n_s^\star}{n^\star} - \frac{1}{2}\left(\frac{n_s^\star}{n^\star}\right)^2\right)$$

Temperature Dependence of α and β

Because n* is small, we are interested in the parameters near T_C

$$H_c(T) = H_{co}\left(1 - \left(\frac{T}{T_c}\right)^2\right) \to 2H_{c0}\left(1 - \frac{T}{T_c}\right) = H_{c,\mathrm{GL}}(0)\left(1 - \frac{T}{T_c}\right)$$

$$n^\star(T) = n_o^\star\left(1 - \left(\frac{T}{T_c}\right)^4\right) \to 4n_0^\star\left(1 - \frac{T}{T_c}\right) = n_{\mathrm{GL}}^\star(0)\left(1 - \frac{T}{T_c}\right)$$

$$\lambda(T) = \frac{\lambda_o}{\sqrt{1-(T/T_c)^4}} \to \frac{\lambda_0/2}{\sqrt{1-(T/T_c)}} = \frac{\lambda_{\mathrm{GL}}(0)}{\sqrt{1-(T/T_c)}}$$

Therefore.

$$\alpha(T) = -\frac{\mu_o H_{c,\mathrm{GL}}^2(0)}{n_{\mathrm{GL}}^\star(0)}\left(1 - \frac{T}{T_c}\right) \qquad \beta = \frac{\mu_o H_{c,\mathrm{GL}}^2(0)}{\left(n_{\mathrm{GL}}^\star(0)\right)^2}$$

Generalization with n_s* (r)

An order parameter (complex) is defined such that

$$\Psi(\mathbf{r}) = \sqrt{n_s^\star(\mathbf{r})}e^{i\theta(\mathbf{r})}$$

Ginzburg and Landau intuited that

$$\begin{aligned} G_A(\vec{\mathcal{H}}, T, \Psi) &= G_n(0,T) \\ &+ \int_{V_s}\left(\alpha|\Psi(\mathbf{r})|^2 + \frac{1}{2}\beta|\Psi(\mathbf{r})|^4 + \frac{1}{2m^\star}\left|\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}\right)\Psi(\mathbf{r})\right|^2\right) dv \\ &+ \frac{1}{2\mu_o}\int_{V_s}\mathrm{B}^2(\mathbf{r})\,dv \,-\, V_s\vec{\mathcal{H}}\cdot\vec{\mathcal{B}}\,. \end{aligned}$$

where $\vec{\mathcal{B}} = \frac{1}{V_s}\int_{V_s}\mathbf{B}(\mathbf{r})\,dv$

Normal State

$$\begin{aligned} G_A(\vec{\mathcal{H}}, T, \Psi) &= G_n(0,T) \\ &+ \int_{V_s}\left(\alpha|\Psi(\mathbf{r})|^2 + \frac{1}{2}\beta|\Psi(\mathbf{r})|^4 + \frac{1}{2m^\star}\left|\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}\right)\Psi(\mathbf{r})\right|^2\right) dv \\ &+ \frac{1}{2\mu_o}\int_{V_s}\mathrm{B}^2(\mathbf{r})\,dv \,-\, V_s\vec{\mathcal{H}}\cdot\vec{\mathcal{B}}\,. \end{aligned}$$

NORMAL STATE $\Psi(\mathbf{r}) = 0$

$$G_n(\vec{\mathcal{H}}, T) = G_n(0,T) + \frac{1}{2\mu_o}\int_{V_s}\mathbf{B}^2\,dv \,-\, V_s\vec{\mathcal{H}}\cdot\vec{\mathcal{B}}$$

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Superconducting State

$$G_A(\vec{\mathcal{H}},T,\Psi) = G_n(0,T) + \int_{V_s}\left(\alpha|\Psi(\mathbf{r})|^2 + \frac{1}{2}\beta|\Psi(\mathbf{r})|^4 + \frac{1}{2m^\star}\left|\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}\right)\Psi(\mathbf{r})\right|^2\right)dv + \frac{1}{2\mu_o}\int_{V_s}\mathrm{B}^2(\mathbf{r})\,dv - V_s\vec{\mathcal{H}}\cdot\vec{\mathcal{B}}.$$

SUPERCONDUCTING STATE $\Psi(\mathbf{r}) = \sqrt{n_s^\star}e^{i\theta(\mathbf{r})}$

$$\mathbf{J}_\mathrm{S} = \frac{q^\star n_s^\star}{m^\star}\left(\hbar\nabla\theta - q^\star\mathbf{A}\right) \quad \& \quad \Lambda_s = \frac{m^\star}{n_s^\star(q^\star)^2}$$

So that

$$G_s(\vec{\mathcal{H}},T) = G_n(0,T) + V_s\left(\alpha(T)n_s^\star + \frac{1}{2}\beta(T)n_s^{\star\,2}\right) + \frac{1}{2\mu_o}\int_{V_s}\left(\mathbf{B}^2 + \mu_o\Lambda_s\mathbf{J}_\mathrm{S}^2\right)dv - V_s\vec{\mathcal{H}}\cdot\vec{\mathcal{B}}$$

Coherence length ξ

Use the definitions of the Ginzburg-Landau parameters to write as

$$G_s(\vec{\mathcal{H}},T) = G_n(0,T) - \mu_o H_c^2\int_{V_s}\left(\frac{n_s^\star}{n^\star} - \frac{1}{2}\left(\frac{n_s^\star}{n^\star}\right)^2 - \xi^2\left(\nabla\sqrt{\frac{n_s^\star}{n^\star}}\right)^2\right)dv + \frac{1}{2\mu_o}\int_{V_s}\left(\mathbf{B}^2 + \mu_o\Lambda\frac{n^\star}{n_s^\star}\mathbf{J}_\mathrm{S}^2\right)dv - V_s\vec{\mathcal{H}}\cdot\vec{\mathcal{B}}.$$

And the coherence length ξ is defined by

$$\xi^2 \equiv -\frac{\hbar^2}{2m^\star\alpha} = \frac{\hbar^2 n^\star}{2m^\star\mu_o H_c^2} = \frac{\Phi_o^2}{8\pi^2\mu_o^2\lambda^2 H_c^2}$$

So that

$$\lim_{T\to T_c}\xi(T) = \frac{\xi_{\mathrm{GL}}(0)}{\sqrt{1-(T/T_c)}} \quad \text{and also} \quad H_c = \frac{\Phi_o}{2\sqrt{2}\pi\mu_o\lambda\xi}$$

The Ginzburg-Landau Equations

$$
\begin{aligned}
G_A(\vec{\mathcal{H}},T) &= G_n(0,T) \\
&+ \int_{V_s}\left(\alpha|\Psi(\mathbf{r})|^2 + \frac{1}{2}\beta|\Psi(\mathbf{r})|^4 + \frac{1}{2m^\star}\left|\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}\right)\Psi(\mathbf{r})\right|^2\right)dv \\
&+ \frac{1}{2\mu_o}\int_{V_s}\mathbf{B}^2(\mathbf{r})\,dv - V_s\vec{\mathcal{H}}\cdot\vec{\mathcal{B}}
\end{aligned}
$$

The calculus of variations finds the wavefunction that minimizes G:

$$
\boxed{\alpha\Psi(\mathbf{r}) + \beta\,|\Psi(\mathbf{r})|^2\,\Psi(\mathbf{r}) + \frac{1}{2m^\star}\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}\right)^2\Psi(\mathbf{r}) = 0}
$$

$$
\mathbf{J}_\mathrm{s} = \frac{q^\star}{m^\star}\,\mathrm{Re}\left\{\Psi^*\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}\right)\Psi\right\}
$$

Comparison of GL with MQM

GL:

$$
\frac{1}{2m^\star}\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}\right)^2\Psi(\mathbf{r}) + \beta\,|\Psi(\mathbf{r})|^2\,\Psi(\mathbf{r}) = -\alpha\Psi(\mathbf{r})
$$

MQM (time independent S-Eqn): $\Psi_{\mathrm{MQM}}(\mathbf{r},t) = \Psi(\mathbf{r})e^{i\theta(\mathbf{r},t)}$

$$
\frac{1}{2m^\star}\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}\right)^2\Psi(\mathbf{r}) + V(\mathbf{r})\Psi(\mathbf{r}) = -\hbar\frac{\partial}{\partial t}\theta(\mathbf{r},t)\,\Psi(\mathbf{r})
$$

GL is the same as MQM when the energy is constant

$$
-\hbar\frac{\partial}{\partial t}\theta(\mathbf{r},t) = \mathcal{E} = -\alpha
$$

and we interpret the internal potential as

$$
V(\mathbf{r}) = \beta|\Psi(\mathbf{r})|^2
$$

Dimensionless Order Parameter

Let $f(\mathbf{r}) = \dfrac{\Psi(\mathbf{r})}{\sqrt{n^\star}}$

And recall that $n^\star = -\dfrac{\alpha}{\beta}$ and $\xi^2 = -\dfrac{\hbar^2}{2m^\star\alpha}$

Then the two GL Equations can be written as:

$$\xi^2 \left(\frac{\nabla}{i} + \frac{2\pi}{\Phi_o}\mathbf{A}\right)^2 f + |f|^2 f - f = 0$$

$$\mathbf{J}_\mathsf{S} = -\frac{\Phi_o}{2\pi\Lambda}\,\mathrm{Re}\left\{f^* \left(\frac{\nabla}{i} + \frac{2\pi}{\Phi_o}\mathbf{A}\right) f\right\}$$

Example: The Depairing Current

Consider the density of electrons to be independent of position, then |f | is a constant but the phase can be a function of position.

$$f = |f|\, e^{i\theta(\mathbf{r})}$$

The current density is then

$$\mathbf{J}_\mathsf{S} = -\frac{\Phi_o}{2\pi\Lambda}\,|f|^2 \left(\nabla\theta + \frac{2\pi}{\Phi_o}\mathbf{A}\right)$$

The GL EQN gives

$$\xi^2 |f| \left(\nabla\theta + \frac{2\pi}{\Phi_o}\mathbf{A}\right)^2 + |f|^3 - |f| = 0$$

Therefore,

$$|\mathbf{J}_\mathsf{S}| = \frac{\Phi_o}{2\pi\Lambda\xi}\,|f|^2 \sqrt{1 - |f|^2}$$

Depairing Current Density

$$|\mathbf{J}_s| = \frac{\Phi_o}{2\pi\Lambda\xi}|f|^2\sqrt{1-|f|^2}$$

$$J_{\text{depair}} = \frac{\Phi_o}{3\sqrt{3}\pi\mu_o\lambda^2\xi}$$

$$= \frac{2\sqrt{2}}{3\sqrt{3}}\frac{H_c}{\lambda}$$

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Please see: Figure 10.2, page 504, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Josephson Microbridge

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Please see: Figure 10.3, page 507, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

$$\xi^2\nabla^2 f + f\left(1-|f|^2\right) = 0$$

$$\xi^2\nabla^2 f \sim \frac{\xi^2}{\ell^2}f \gg 1 \quad \Rightarrow \quad \nabla^2 f = 0$$

Laplace's Equation

Josephson Microbridge

Let

$$f(\mathbf{r}) = |f_1| e^{i\theta_1} g(\mathbf{r}) + |f_2| e^{i\theta_2} [1 - g(\mathbf{r})]$$

where $\nabla^2 g(\mathbf{r}) = 0$

$$g(\mathbf{r}) = \begin{cases} 1 & \text{as } x \rightarrow -\infty \\ 0 & \text{as } x \rightarrow +\infty \end{cases} \qquad \mathbf{n} \cdot \nabla g = 0$$

$$\mathbf{J}_\mathsf{s} = -\frac{\Phi_o}{2\pi\Lambda} \mathrm{Re}\left\{\frac{1}{i} f^* \nabla f\right\} \Rightarrow \mathbf{J}_\mathsf{s} = \underbrace{\frac{\Phi_o |f_1||f_2|}{2\pi\Lambda} \nabla g}_{\mathbf{J_C}} \sin(\theta_1 - \theta_2)$$

Type I vs Type II

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Please see: Figure 10.4, page 510, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

$$\sigma_{\mathcal{E}} = \frac{\Delta G}{S} \approx \frac{1}{2} \mu_o H_c^2 (\xi - \lambda)$$

Condensation enery vs magnetic energy

$\xi > \lambda$ type I and $\xi < \lambda$ type II