MagnetoQuasiStatics Lecture 3

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Outline

1. **Magnetoquasistatic Equations**
2. **Magnetic Diffusion Equation**
3. **Examples**
   - **A. Infinite Slab**
     - **i. Poorly conducting regime**
     - **ii. Perfectly conducting regime**
   - **B. Sphere : Magnetic scalar potential**
   - **C. Cylinder: Current biased solutions**

MagnetoQuasiStatics

$$\left.\begin{aligned}\nabla \times \mathbf{H} &= \mathbf{J}\\ \nabla \cdot \mathbf{B} &= 0\\ \nabla \cdot \mathbf{J} &= 0\end{aligned}\right\}$$ Solve first

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Please see: Figure 2.9, page 34, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ Solve for **E** once **B** is found

Boundary conditions:

$$n \times (H_2 - H_1) = K \qquad n \cdot (B_2 - B_1) = 0 \qquad n \cdot (J_2 - J_1) = 0$$

MQS: Magnetic Diffusion Equation

For a metal **B** = μ_0 **H**, **D** = ε_0 **E** and **J** = σ_0**E**, so that

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$$\left(\mu \sigma_o \frac{\partial}{\partial t} - \nabla^2\right) \mathbf{H} = 0$$

Magnetic Diffusion Equation

Infinite Slab

Let $\mathbf{H}(\mathbf{r},t) = \mathrm{Re}\left\{\hat{H}(y)\, e^{j\omega t}\right\}\mathbf{i}_z$

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Please see: Figure 2.24, page 61, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity* Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Therefore,

$$\left(j\omega\mu\sigma_o - \frac{d^2}{dy^2}\right)\hat{H}(y) = 0$$

and

$$\hat{H}(y) = C\cosh ky + D\sinh ky$$

$$\mathbf{H}_{\mathrm{app}} = \mathrm{Re}\left\{\hat{H}_o\, e^{j\omega t}\right\}\mathbf{i}_z$$

so that

$$\left(\mu\sigma_o\frac{\partial}{\partial t} - \nabla^2\right)\mathbf{H} = 0$$

$$k^2 = j\omega\mu\sigma_o$$

General Solution

$$k^2 = j\omega\mu\sigma_o \quad \Rightarrow \quad k = \frac{(1+j)}{\delta}$$

where the *magnetic diffusion length* is $\delta \equiv \sqrt{\dfrac{2}{\omega\mu\sigma_o}}$

Boundary Conditions demand D=0, so that $\hat{H}(y) = C\cosh ky$

Boundary Conditions demand

$$H_z(a) = H_z(-a) = C\cosh ka = \hat{H}_o$$

Therefore, $$\mathbf{H} = \mathrm{Re}\left\{\hat{H}_o\frac{\cosh ky}{\cosh ka}\, e^{j\omega t}\right\}\mathbf{i}_z$$

Fields and Currents for |y|< a

$$\mathbf{H} = \text{Re}\left\{\hat{H}_o \frac{\cosh ky}{\cosh ka} e^{j\omega t}\right\} \mathbf{i}_z \qquad \mathbf{J} = \text{Re}\left\{\hat{H}_o k \frac{\sinh ky}{\cosh ka} e^{j\omega t}\right\} \mathbf{i}_x$$

Poor conductor limit
Thin film limit

$$a \ll \delta \iff \omega\tau_m \ll 1$$

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Please see: Figure 2.14, page 45, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Perfect conductor limit
Bulk limit

$$a \gg \delta \iff \omega\tau_m \gg 1$$

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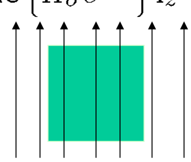
Fields and Currents for |y|< a

$$\mathbf{H} = \text{Re}\left\{\hat{H}_o \frac{\cosh ky}{\cosh ka} e^{j\omega t}\right\} \mathbf{i}_z \qquad \mathbf{J} = \text{Re}\left\{\hat{H}_o k \frac{\sinh ky}{\cosh ka} e^{j\omega t}\right\} \mathbf{i}_x$$

Poor conductor limit
Thin film limit

$$a \ll \delta \iff \omega\tau_m \ll 1$$

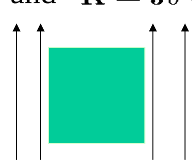
$$\mathbf{H} = \text{Re}\left\{\hat{H}_o\, e^{j\omega t}\right\} \mathbf{i}_z \quad \text{and} \quad \mathbf{J} = 0$$



Perfect conductor limit
Bulk limit

$$a \gg \delta \iff \omega\tau_m \gg 1$$

$$\mathbf{H} = 0 \quad \text{and} \quad \mathbf{K} = \mathbf{J}\delta = \text{Re}\left\{\hat{H}_o\, e^{j\omega t}\right\} \mathbf{i}_x$$



Sphere in a magnetic field

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Please see: Figure 2.16, page 47, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Poorly conducting regime	Perfectly conducting regime
$\delta \gg R$	$\delta \ll R$
$\omega\tau_m \ll 1$	$\omega\tau_m \gg 1$

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$$\mathbf{H}_{\mathrm{app}} = \mathrm{Re}\left\{\widehat{H}_o\, e^{j\omega t}\right\} \mathbf{i}_z$$

$$\left(\mu\sigma_o \frac{\partial}{\partial t} - \nabla^2\right) \mathbf{H} = 0$$

$$\mathbf{H}_{\mathrm{app}} = \mathrm{Re}\left\{\widehat{H}_o\, e^{j\omega t}\right\} \mathbf{i}_z$$

$$\mathbf{K}(r = R) = 0$$

$$\begin{aligned}\mathbf{H}(r \geq R) &= \mathrm{Re}\left\{\widehat{H}_o\left(1 - \left(\frac{R}{r}\right)^3\right)\cos\theta\, e^{j\omega t}\right\}\mathbf{i}_r \\ &\quad - \mathrm{Re}\left\{\widehat{H}_o\left(1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right)\sin\theta\, e^{j\omega t}\right\}\mathbf{i}_\theta .\end{aligned}$$

$$\mathbf{K}(r = R) = -\mathrm{Re}\left\{\frac{3}{2}\widehat{H}_o \sin\theta\, e^{j\omega t}\right\}\mathbf{i}_\phi$$

Sphere in the perfectly conducting regime

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In this bulk approximation, there is no current density **J** = 0, only a surface current **K.** Therefore, in all space

$$\nabla \times \mathbf{H} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

Define a *magnetic scalar potential*

$$\mathbf{H} \equiv -\nabla\psi$$

which then satisfies Laplace's equation

$$\nabla \cdot \nabla\psi = \nabla^2\psi = 0$$

Solutions to Laplace's Equation

$\mathbf{H} \equiv -\nabla\psi$

	Spherical	Cylindrical
Uniform field	$\psi \propto r\cos\theta = z$	$\psi \propto r\cos\theta = x$ $\psi \propto z$
Monopole field	$\psi \propto \dfrac{1}{r}$	$\psi \propto \ln r$
Dipole field	$\psi \propto \dfrac{\cos\theta}{r^2}$	$\psi \propto \dfrac{\cos\theta}{r}$

Sphere in a magnetic field

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Potential for the uniform field:

$$\psi_o = \mathrm{Re}\left\{-\widehat{H}_o r\cos\theta e^{j\omega t}\right\} \equiv \mathrm{Re}\left\{\widehat{\psi}_o e^{j\omega t}\right\}$$

To have **H** = 0 for r < R, the surface current **K** must produce a potential such that

$$\begin{aligned}\widehat{\psi}_{\text{in}}(r \le R) &= \widehat{\psi}_o(r \le R) + \widehat{\psi}_K(r \le R) \\ &= -\widehat{H}_o r\cos\theta + C_1 r\cos\theta\end{aligned}$$

$$\mathrm{H}_{\text{app}} = \mathrm{Re}\left\{\widehat{H}_o e^{j\omega t}\right\}\mathrm{i}_z$$

$$\mathbf{H} \equiv -\nabla\psi$$

$$\nabla^2\psi = 0$$

To match the boundary conditions for all angles, a dipole field is needed on the outside

$$\begin{aligned}\widehat{\psi}_{\text{out}}(r \ge R) &= \widehat{\psi}_o(r \ge R) + \widehat{\psi}_K(r \ge R) \\ &= -\widehat{H}_o r\cos\theta + C_2(\cos\theta/r^2)\end{aligned}$$

Inside the Sphere

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$$\begin{aligned}\widehat{\psi}_{\text{in}}(r \le R) &= \widehat{\psi}_o(r \le R) + \widehat{\psi}_K(r \le R) \\ &= -\widehat{H}_o r \cos\theta + C_1 r \cos\theta\end{aligned}$$

Therefore, $C_1 = \widehat{H}_o$

Outside the Sphere

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$$\begin{aligned}\widehat{\psi}_{\text{out}}(r \ge R) &= \widehat{\psi}_o(r \ge R) + \widehat{\psi}_K(r \ge R) \\ &= -\widehat{H}_o r \cos\theta + C_2(\cos\theta / r^2)\end{aligned}$$

Use the boundary condition

$$\mathbf{i_r} \cdot \left(-\mu_o \nabla \widehat{\psi}_{\text{out}} + \mu \nabla \widehat{\psi}_{\text{in}}\right)\Big|_{r=R} = 0$$

Therefore, $C_2 = -\frac{1}{2} R^3 \widehat{H}_o$

Outside the Sphere

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$$\begin{aligned}\widehat{\psi}_{\text{out}}(r \geq R) &= \widehat{\psi}_o(r \geq R) + \widehat{\psi}_K(r \geq R) \\ &= -\widehat{H}_o r \cos\theta \; - \frac{1}{2} R^3 \, \widehat{H}_o (\cos\theta / r^2)\end{aligned}$$

$$\begin{aligned}\mathbf{H}(r \geq R) &= \text{Re}\left\{ \widehat{H}_o \left(1 - \left(\frac{R}{r}\right)^3 \right) \cos\theta \, e^{j\omega t} \right\} \mathbf{i}_r \\ &\quad - \text{Re}\left\{ \widehat{H}_o \left(1 + \frac{1}{2} \left(\frac{R}{r}\right)^3 \right) \sin\theta \, e^{j\omega t} \right\} \mathbf{i}_\theta \,.\end{aligned}$$

$$\mathbf{K}(r = R) = -\text{Re}\left\{ \frac{3}{2} \, \widehat{H}_o \sin\theta \, e^{j\omega t} \right\} \mathbf{i}_\phi$$

Current along a cylinder

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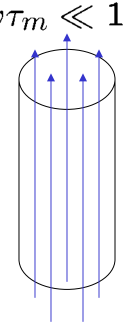
$$\mathbf{I} = \text{Re}\left\{ \widehat{I}_o \, e^{j\omega t} \right\} \mathbf{i}_z$$

Poorly conducting regime

$$\delta \gg R$$

$$\omega \tau_m \ll 1$$

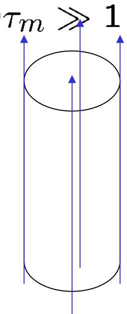


$$\mathbf{J}(r \leq R) = \frac{\mathbf{I}}{\pi R^2}$$

Perfectly conducting regime

$$\delta \ll R$$

$$\omega \tau_m \gg 1$$



$$\mathbf{J}(r \leq R) = 0$$

$$\mathbf{K}(r = R) = \frac{\mathbf{I}}{2\pi R}$$

Current along a cylinder: poor conductor

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$$\mathbf{I} = \mathrm{Re}\left\{ \widehat{I}_o \, e^{j\omega t} \right\} \mathbf{i}_z$$

$$\mathbf{J}(r \le R) = \frac{\mathbf{I}}{\pi R^2}$$

The fields from Ampere's law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s}$$

Inside: $H\,2\pi r = \dfrac{I}{\pi R^2}\,\pi r^2$

$$\mathbf{H}(r \le R) = \mathrm{Re}\left\{ \frac{\widehat{I}_o}{2\pi R}\,\frac{r}{R}\,e^{j\omega t} \right\} \mathbf{i}_\phi$$

Outside: $H\,2\pi r = \dfrac{I}{\pi R^2}\,\pi R^2$

$$\mathbf{H}(r \ge R) = \mathrm{Re}\left\{ \frac{\widehat{I}_o}{2\pi r}\,e^{j\omega t} \right\} \mathbf{i}_\phi$$

Therefore, $\mathrm{K}(r = R) = 0$

Current along a cylinder: perfect conductor

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$$\mathbf{I} = \mathrm{Re}\left\{ \widehat{I}_o \, e^{j\omega t} \right\} \mathbf{i}_z$$

$$\mathbf{J}(r \le R) = \frac{\mathbf{I}}{\pi R^2}$$

The fields from Ampere's law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s}$$

Inside: $H\,2\pi r = 0$

$$\mathbf{H}(r \le R) = 0$$

Outside: $H\,2\pi r = I$

$$\mathbf{H}(r \ge R) = \mathrm{Re}\left\{ \frac{\widehat{I}_o}{2\pi r}\,e^{j\omega t} \right\} \mathbf{i}_\phi$$

Therefore, $\mathrm{K}(r = R) = \dfrac{\mathrm{I}}{2\pi R}$