Lecture 8: Perfect Diamagnetism

Outline

1. **Description of a Perfect Diamagnet**
   - **Method I and Method II**
   - **Examples**

2. **Energy and Coenergy in Methods I and II**

3. **Levitating magnets and Maglev trains**

October 4, 2005

Description of Perfect Diamagetism

Image removed for copyright reasons.
Please see: Figure 2.17, page 50, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Surface currents or internal induced magnetization?

Methods I and II

Image removed for copyright reasons.
Please see: Figure 4.10, page 173, from Orlando, T., and K. Delin.
Foundations of Applied Superconductivity. Reading, MA:
Addison-Wesley, 1991. ISBN: 0201183234.

$$\mathbf{B} = \mu_o \mathbf{H}^I \qquad \mathbf{B} = \mu_o \left(\mathbf{H}^{II} + \mathbf{M} \right)$$

Method I　　Method II

B field is the same in both methods.

Example: Magnetized Sphere

Image removed for copyright reasons.
Please see: Figure 4.11, page 174, from Orlando, T., and K. Delin.
Foundations of Applied Superconductivity. Reading, MA:
Addison-Wesley, 1991. ISBN: 0201183234.

$$\mathbf{M} = M_o \, \mathbf{i}_z = M_o \left(\cos\theta \, \mathbf{i}_r - \sin\theta \, \mathbf{i}_\theta \right) \qquad \text{for } \ r \leq R.$$

$$\nabla \times \mathbf{H} = \mathbf{J} = 0 \quad \Rightarrow \quad \mathbf{H} = -\nabla \psi$$

$$\left. \begin{array}{c} \nabla \cdot \mathbf{B} = 0 \\ \mathbf{B} = \mu_o \left(\mathbf{H} + \mathbf{M} \right) \end{array} \right\} \quad \Rightarrow \quad \nabla^2 \psi = \nabla \cdot \mathbf{M}$$

For this example: $\nabla^2 \psi = 0$ Laplace's equation

Massachusetts Institute of Technology
6.763 2005 Lecture 8

Boundary Conditions

Inside the sphere:

$$\psi(r \le R) = C_1 \, r \cos\theta \;\Rightarrow\; \mathbf{H}(r \le R) = -C_1 \left(\cos\theta \, \mathbf{i}_r - \sin\theta \, \mathbf{i}_\theta\right)$$

Outside the sphere:

$$\psi(r \ge R) = C_2 \frac{\cos\theta}{r^2} \;\Rightarrow\; \mathbf{H}(r \ge R) = \frac{C_2}{r^3}\left(2\cos\theta \, \mathbf{i}_r + \sin\theta \, \mathbf{i}_\theta\right)$$

Boundary Conditions:

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K} \;\Rightarrow\; C_1 R^3 = C_2$$

$$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \;\Rightarrow\; \mathbf{i}_r \cdot \left(\mathbf{H}|_{r=R^+} - \mathbf{H}|_{r=R^-}\right) = \mathbf{i}_r \cdot \mathbf{M}|_{r=R^-}$$

Therefore, $\mathbf{M} = M_o \left(\cos\theta \, \mathbf{i}_r - \sin\theta \, \mathbf{i}_\theta\right) \;\Rightarrow\; C_1 = \dfrac{M_o}{3}$

Magnetized Sphere

$$\mathbf{H}(r \le R) = -\frac{M_o}{3}\mathbf{i}_z = -\frac{M_o}{3}\left(\cos\theta \, \mathbf{i}_r - \sin\theta \, \mathbf{i}_\theta\right)$$

$$\mathbf{H}(r \ge R) = \frac{M_o}{3}\left(\frac{R}{r}\right)^3 \left(2\cos\theta \, \mathbf{i}_r + \sin\theta \, \mathbf{i}_\theta\right)$$

Magnetized Sphere in field

Image removed for copyright reasons.
Please see: Figure 4.11, page 174, from Orlando, T., and K. Delin *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

$$\mathbf{H}(r \le R) = \left(H_o - \frac{M_o}{3}\right)\mathbf{i}_z$$

$$\mathbf{H}_{\mathsf{app}} = H_o\,\mathbf{i}_z$$

$$\begin{aligned}\mathbf{H}(r \ge R) &= \left(H_o + \frac{2}{3}M_o\left(\frac{R}{r}\right)^3\right)\cos\theta\,\mathbf{i}_r \\ &\quad - \left(H_o - \frac{1}{3}M_o\left(\frac{R}{r}\right)^3\right)\sin\theta\,\mathbf{i}_\theta\end{aligned}$$

For this to describe a superconductor (bulk limit), then B=0 inside.

Therefore, $0 = \mathbf{M} + \mathbf{H}(r \le R) = M_o\mathbf{i}_z + \left(H_o - \frac{M_o}{3}\right)\mathbf{i}_z$

So that $M = -\frac{3}{2}H_o$

Comparison of Methods

Method I

$$\mathbf{B}^I = \mu_o\mathbf{H}^I$$

Method II

$$\mathbf{B} = \mu_o\left(\mathbf{H}^{II} + \mathbf{M}\right)$$

Inside a bulk superconducting sphere:

Method I:

$$\mathbf{B}^I = 0$$

$$\mathbf{H}^I = 0$$

$$\mathbf{K}^I = -\mathsf{Re}\left\{\frac{3}{2}\hat{H}_o\sin\theta\right\}\mathbf{i}_\phi$$

Method II:

$$\mathbf{B}^{II} = 0$$

$$\mathbf{H}^{II} = \frac{3}{2}H_o\mathbf{i}_z$$

$$\mathbf{M} = -\frac{3}{2}H_o\mathbf{i}_z$$

$$\mathbf{K}^{II} = 0$$

B field is the same, but not **H**.

Methods I and II: Summary

Maxwell's Equations

$$\nabla \cdot \mathbf{B} = 0$$

Method I

$$\nabla \times \mathbf{H}^I = \mathbf{J}^I$$

$$\mathbf{B} = \mu_o \mathbf{H}^I$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J}^I$$

$$\mathbf{J}^\mathrm{I} = \underbrace{\left(\mathbf{J}_\mathsf{s,app} + \mathbf{J}_\mathsf{s,ind}\right)}_{\mathbf{J}_\mathsf{s}^\mathrm{I}} + \mathbf{J}_\mathsf{n}$$

Method II

$$\nabla \times \mathbf{H}^{II} = \mathbf{J}^{II}$$

$$\mathbf{B} = \mu_o \left(\mathbf{H}^{II} + \mathbf{M}\right)$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J}^{II} + \mu_o \underbrace{\nabla \times \mathbf{M}}_{\mathbf{J}_\mathsf{s,ind}}$$

$$\mathbf{J}^\mathrm{II} = \mathbf{J}_\mathsf{s,app} + \mathbf{J}_\mathsf{n}$$

London Equations

Method I

$$\mathbf{E} = \frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_\mathsf{s}^\mathrm{I}\right)$$

$$\nabla \times \left(\Lambda \mathbf{J}_\mathsf{s}^\mathrm{I}\right) = -\mathbf{B}$$

Method II

$$\mathbf{E} = \frac{\partial}{\partial t}\left(\Lambda \mathbf{J}_\mathsf{s}^\mathrm{II}\right) + \frac{\partial}{\partial t}\left(\Lambda \left(\nabla \times \mathbf{M}\right)\right)$$

$$\nabla \times \left(\Lambda \mathbf{J}_\mathsf{s}^\mathrm{II}\right) + \nabla \times \left(\Lambda \left(\nabla \times \mathbf{M}\right)\right) = -\mathbf{B}$$

Why Method II

Question:
The constitutive relations and London's Equations have gotten much more difficult. So why do Method II?

Answer:
The Energy and Thermodynamics are easier, especially when there is no applied current.

So we will find the energy stored in both methods.

Poynting's theorem is a result of Maxwell's equation, so both methods give

$$-\oint_\Sigma (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s} = \int_V \left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\mathbf{J}\right) dv$$

Method I: The Energy

Combining the constitutive relations with Poynting's theorem,

$$-\oint_{\Sigma}(\mathbf{E}\times\mathbf{H}^{\mathrm{I}})\cdot d\mathbf{s} = \underbrace{\int_V \left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}^{\mathrm{I}}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\cdot\frac{\partial}{\partial t}\left(\Lambda\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right)\right) dv}_{\text{Power dW/dt in the E\&M field}} + \int_V \mathbf{E}\cdot\mathbf{J}_{\mathsf{n}}\, dv$$

The energy stored in the electromagnetic field is

$$dW = \int_V \left(\mathbf{E}\cdot d\mathbf{D} + \mathbf{H}^{\mathrm{I}}\cdot d\mathbf{B} + \mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\cdot d\left(\Lambda\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right)\right) dv$$

So that the energy W is a function of **D**, **B**, and $\Lambda\mathbf{J}_s^{I} = \mathbf{v}_s$. However, one rarely has control over these variables, but rather over their conjugates **E**, $\mathbf{H}^I$, and $\mathbf{J}_s^I$.

Method I: The Coenergy

The coenergy is a function of **E**, $\mathbf{H}^I$, and $\mathbf{J}_s^I$ is defined by

$$W + \widetilde{W} = \int_V \left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}^{\mathrm{I}}\cdot\mathbf{B} + \mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\cdot\left(\Lambda\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right)\right) dv$$

and with

$$dW = \int_V \left(\mathbf{E}\cdot d\mathbf{D} + \mathbf{H}^{\mathrm{I}}\cdot d\mathbf{B} + \mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\cdot d\left(\Lambda\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right)\right) dv$$

gives

← EQS → ←——— MQS ———→

$$d\widetilde{W} = \int_V \left(\mathbf{D}\cdot d\mathbf{E} + \mathbf{B}\cdot d\mathbf{H}^{\mathrm{I}} + \Lambda\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\cdot d\mathbf{J}_{\mathsf{s}}^{\mathrm{I}}\right) dv$$

(The coenergy is the Free Energy at zero temperature.)

Interpretation of the Coenergy

Consider the case where there are only magnetic fields:

$$\widetilde{W} = \int_{\mathbf{H}^{\mathrm{I}}} d\widetilde{W} = \int_V \mathbf{B} \cdot d\mathbf{H}^{\mathrm{I}}\, dv \qquad W = \int_{\mathbf{B}} dW = \int_{\mathbf{B}} \int_V \mathbf{H}^{\mathrm{I}} \cdot d\mathbf{B}\, dv$$

Image removed for copyright reasons.
Please see: Figure 4.12, page 184, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

$$W_m + \widetilde{W}_m = H_f B_f$$

The energy and coenergy contain the same information.

Method II

$$dW = \int_V \left(\mathbf{E} \cdot d\mathbf{D} + \mathbf{H}^{\mathrm{II}} \cdot d\mathbf{B} + \mathbf{J}_{\mathsf{s}}^{\mathrm{II}} \cdot d\left(\Lambda \mathbf{J}_{\mathsf{s}}^{\mathrm{II}} + \Lambda \nabla \times \mathbf{M}\right)\right) dv$$

$$d\widetilde{W} = \int_V \left(\mathbf{D} \cdot d\mathbf{E} + \mathbf{B} \cdot d\mathbf{H}^{\mathrm{II}} + \left(\Lambda \mathbf{J}_{\mathsf{s}}^{\mathrm{II}} + \Lambda \nabla \times \mathbf{M}\right) \cdot d\mathbf{J}_{\mathsf{s}}^{\mathrm{II}}\right) dv$$

In the important case when of the MQS limit and $\mathbf{J}_{\mathsf{s}}^{\mathrm{II}} = \mathbf{J}_{\mathsf{s,app}} = 0$

$$dW\big|_{\mathbf{J}_{\mathsf{s,app}}=0} = \int_V \left(\mathbf{H}^{\mathrm{II}} \cdot d\mathbf{B}\right) dv$$

$$d\widetilde{W}\Big|_{\mathbf{J}_{\mathsf{s,app}}=0} = \int_V \left(\mathbf{B} \cdot d\mathbf{H}^{\mathrm{II}}\right) dv$$

Note that these two relations apply also in free space.

Example: Energy of a Superconducting Sphere

Image removed for copyright reasons.
Please see: Figure 2.18, page 50, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

$$\mathbf{B}(r \leq R) = 0 \qquad \mathbf{H}^{II}(r < R) = \frac{3}{2} H_o \mathbf{i}_z$$

$$\mathbf{H}(r \geq R) = \mathrm{Re}\left\{\widehat{H}_o\left(1 - \left(\frac{R}{r}\right)^3\right)\cos\theta\right\}\mathbf{i}_r - \mathrm{Re}\left\{\widehat{H}_o\left(1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right)\sin\theta\right\}\mathbf{i}_\theta$$

$$W_{inside} = \int_B \int_V \left(\mathbf{H}^{\mathrm{II}} \cdot d\mathbf{B}\right) dv = 0$$

$$W_{outside} = \int_{\mathbf{B}} \int_V \left(\mathbf{H}^{\mathrm{II}} \cdot d\mathbf{B}\right) dv = \int_V \frac{1}{\mu_o} \mathbf{B}^2 \, dv \neq 0$$

Magnetic Levitation

Image removed for copyright reasons.
Please see: Figure 4.17, page 195, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Image removed for copyright reasons.
Please see: Figure 4.18, page 195, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Magnetic vs. Gravitational Forces

Image removed for copyright reasons.
Please see: Figure 4.17, page 195, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Image removed for copyright reasons.
Please see: Figure 4.19, page 196, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Magnetic Levitation Equilibrium Point

Image removed for copyright reasons.
Please see: Figure 4.20, page 198, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Image removed for copyright reasons.
Please see: Figure 4.21, page 199, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

$$\Phi = \pi R^2 B_o = 2\pi R \eta B$$

$$W_m = \int_V \frac{1}{\mu_o} B^2\, dv \approx \frac{\pi R^2 \eta}{\mu_o} B^2 = \frac{\Phi^2}{8\pi \mu_o \eta}$$

$$f_m = -\frac{\partial}{\partial \eta} W_m(\Phi, \eta) \approx \frac{\Phi^2}{8\pi \mu_o \eta^2}$$

Force is upwards

Equilibrium point $f_m(\eta_0) = mg$ ⇒ $\eta_o = \sqrt{\dfrac{\Phi^2}{8\pi \mu_o m g}}$

Levitating magnets and trains

$$f_m \approx \frac{\Phi^2}{8\pi\mu_o\eta^2}$$

B = 1 Tesla
Area = 0.5 cm^2
η_0 = 1 cm
Force = 1 Newton

Enough to lift magnet, but not a train

B = 2 Tesla
Area = 100 cm^2
η_0 = 10 cm
Force = 1,200 Newtons

Enough to lift a train

Maglev Train

Magnets on the train are superconducting magnets; the rails are ohmic!

Principle of Maglev

The train travels at a velocity U, and the moving flux lines and the rails "see" a moving magnetic field at a frequency of

$$\omega \sim U/R.$$

If this frequency is much larger than the inverse of the magnetic diffusion time,

$$\tau_m = \mu \sigma_o R d$$

then the flux lines are "repelled" from the ohmic rails.

$$\omega \tau_m \gg 1 \qquad\qquad U \gg \frac{1}{\mu \sigma_o d}$$

From the previous numbers, U > 40 km/hr for levitation.

Real trains have wheels and high-voltage rails

Image removed for copyright reasons.
Please see: Figure 4.23, page 203, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Synchronous motor action down the rails provides thrust to accelerate train to the needed velocity to levitate, and provides a source of energy to further accelerate the train and to overcome the losses due to drag from the wind.

Our Approach to Superconductivity