6.777J/2.732J Design and Fabrication of Microelectromechanical Devices
Spring Term 2007

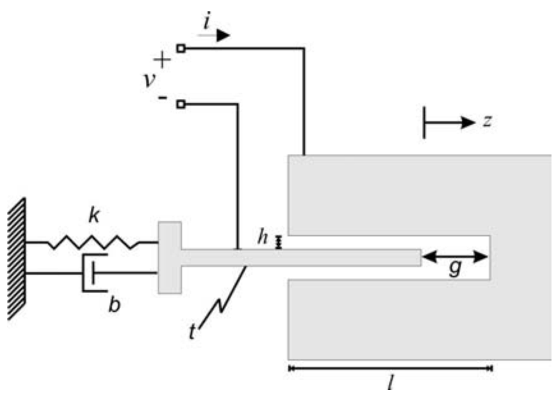
Massachusetts Institute of Technology

PROBLEM SET 4 – SOLUTIONS (25 pts)

Issued 3/7/07 Due 3/16/07

Problem 6.5 (5 pts): The in-plane interdigitated electrostatic (or comb drive) transducer

a). Current drive case



The gap g between the finger tip and the electrode can be expressed in terms of the displacement z as follow,

$$g = g_0 - z$$

The effective capacitor length is hence, $l - g = l - g_0 + z$. The capacitance of the upper part of comb is (by neglecting the electric fields around the finger tip),

$$C_{upper} = \frac{\varepsilon_0 (l-g)t}{h}$$

Since the one-finger model is equivalent to two capacitors in parallel (upper and lower parts), the total capacitance is multiplied by two,

$$C = \frac{2\varepsilon_0 (l-g)t}{h}$$

The energy stored in the system is

$$W = \int_0^Q v dQ = \int_0^Q \frac{Q}{C} dQ = \frac{Q^2}{2C}$$

Substitute C into the equation above, we have

$$W = \frac{hQ^2}{4\varepsilon_0 (l-g)t}$$

The force becomes

$$F = \left(\frac{\partial W}{\partial g}\right)_Q = \frac{Q^2}{4} \frac{h}{\varepsilon_0 t (l-g)^2}$$

To find the gap, we have

$$F = kz = -k(g - g_0)$$

$$\Rightarrow g = g_0 - \frac{F}{k} = g_0 - \frac{Q^2 h}{4\varepsilon_0 t k (l-g)^2}$$

The voltage can be expressed as,

$$v = \left(\frac{\partial W}{\partial Q}\right)_g = \frac{hQ}{2\varepsilon_0 t (l-g)}$$

b). Assumptions made in calculating the force in a).

a. The upper and lower gaps of the comb fingers are exactly the same as fabricated
b. Fringing effects are neglected.
c. The energy stored between the finger tip and the electrode (gap g) is neglected.
d. The electrical permittivity of air is assumed to be the one of free space.
e. There is no space charge in the space between the plates.
f. The actuation of the comb drive is quasi-static.
g. Charge distributed uniformly on the plates

c). Voltage drive case

We can use co-energy to find force and charge.

The co-energy in the capacitor is

$$W^*(v,g) = \int_0^V Q dv = \int_0^V C v dv = \frac{Cv^2}{2} = \frac{\varepsilon_0 (l-g) t v^2}{h}$$

The force

$$F = -\left(\frac{\partial W^*}{\partial g}\right)_V$$

$$F = \frac{\varepsilon_0 t v^2}{h}$$

And the charge

$$Q = \left(\frac{\partial W^*}{\partial v}\right)_g$$

$$Q = \frac{2\varepsilon_0 (l-g) t v}{h}$$

To find the gap, we again have

$$F = kz = -k(g - g_0)$$

$$\frac{\varepsilon_0 t v^2}{h} = -k(g - g_0)$$

$$g = g_0 - \frac{\varepsilon_0 t v^2}{kh}$$

d). The net force for the voltage drive case:

$$F_{net} = -\frac{\varepsilon_0 t v^2}{h} + K(g_0 - g)$$

$$\frac{\partial F_{net}}{\partial g} = -k < 0$$

The effective spring constant is a negative constant, which means the increase in gap will cause the decrease in force, therefore, the system is always stable and there is no spring softening/hardening.

e). The net force for the current drive case:

$$F_{net} = -\frac{Q^2}{4}\frac{h}{\varepsilon_0 t (l-g)^2} + k(g_0 - g)$$

$$\frac{\partial F_{net}}{\partial g} = k_{eff} = -\frac{Q^2 h}{2\varepsilon_0 t (l-g)^3} - k < 0$$

The effective spring constant is also negative, however, it's not a constant. Since g always decreases from g_0 to 0 when actuation starts, $|k_{eff}|$ decreases as the comb drive is actuated, creating the spring softening effect. However, since k_{eff} is always negative regardless of the value of g, the system is always stable, and hence, no pull-in will occur. This conclusion is based on the assumption that the electric fields around the fingertips are negligible. Also, it is assumed that the upper gap and lower gap of the comb fingers are the same, while in reality, they might vary due to nonuniform etching or other fabrication effects. Pull-in, hence, can occur due to these secondary effects.

Problem 6.9 (2 pts): Design the spring

Since we have 2 springs in parallel (as shown in the figure), the effective spring constant of each is $k_{eq} = k_{parallel}/2 = 0.5\ kN/m$

Each spring acts as a beam with a fixed support at one end (the anchor) and subjected at the other end to a zero slope boundary condition (because of its connection to the mass), despite its ability to translate. This BC imposes a moment reaction at the other end. Hence the resultant deflection is the superposition of 2 tabulated deflections: that of a cantilever beam subjected to a point force at its end and that of a cantilever beam subjected to a moment at its end. The moment is unknown yet, and will be found by substituting the zero-slope BC into the resultant deflection profile.

For a cantilever (length L, width a, thickness t) with point force F at the end: $\hat{w}_1(x) = \frac{2F}{Eta^3}(-x^3 + 3x^2 L)$

For a cantilever (length L, width a, thickness t) with a moment M at the end: $\hat{w}_2(x) = \frac{6M}{Eta^3}x^2$

The resulting deflection is thus: $\hat{w}(x) = \hat{w}_1(x) + \hat{w}_2(x) = \frac{2F}{Eta^3}(-x^3 + 3x^2 L) + \frac{6M}{Eta^3}x^2$

Using the boundary condition,

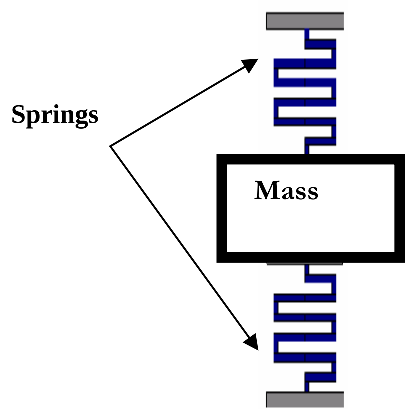
$$\left.\frac{d\hat{w}(x)}{dx}\right|_{x=L} = \left(\frac{2F}{Eta^3}(-3x^2 + 6xL) + \frac{12M}{Eta^3}x\right)_{x=L} = 0$$

we can solve for M and find:

$$M = -\frac{FL}{2}$$

We then plug back into our deflection equation to find:

$$\hat{w}(x) = \frac{2F}{Eta^3}(-x^3 + 1.5x^2 L)$$



Evaluating at x=L provides:

$$\hat{w}(L) = \frac{FL^3}{Eta^3}$$

We can then solve for the equivalent spring constant.

$$k_{eq} = \frac{F}{\hat{w}(L)} = \frac{Eta^3}{L^3}$$

Subsituting $E \approx 150$ GPa, $a = 10$ µm, and $t = 200$ µm, we get: $L = 310.7$ µm.
We can fit such a long spring into a minimal wafer area by folding it, as shown:

Problem 6.10 (7 pts): Design a simple switch

(a) Using t, l, w , and g in [m] to avoid confusion, and all other parameters in standard SI units:

- $V_{PI} = \sqrt{\dfrac{8kg^3}{27\varepsilon A_{cap}}} = 10V$ **-------(1)**

  (assuming electrical force applied as point force at tip)
- $k = \dfrac{Ewt^3}{4l^3}$ **------------(2)**

  (assuming ideal cantilever support)
- $A_{cap} = l_o w$ **------------(3)**

  (assuming capacitor area does not vary much as switch is deflected)
- $R = \dfrac{1}{\sigma}\dfrac{l}{wt} \le 10\Omega$ **----------(4)**

  (assuming changes in length and width of cantilever between the closed and open switch modes are negligible)
- $l \ge 5w$ **------------(5)**
- $w \ge 10t$ **-----------(6)**
- Cost constraint: $lw \times 200 \times 10^6 \$ / m^2 + (t - 2\times 10^{-6}) \times u(t - 2\times 10^{-6} m) \times 2\times 10^6 \$ / m \le 1\$$ **-------(7)**

  (u is the unit step function, since we start paying for the thickness once it exceeds 2 μm, according to the problem statement).

(b) Substituting E = 150 GPa, σ = 10^5 S/m, ρ=2300 kg/m ^{3}, l_0 = 10 μm, and $\varepsilon = \varepsilon_0$ = 8.854 x 10^{-12} F/m (i.e. assuming the gap is vacuum), we get:

$\dfrac{gt}{l} = 9.2712 \times 10^{-9} m$ --------- (A) (from (1), (2), and (3))

$\dfrac{l}{wt} \le 10^6 m^{-1}$ ----------(B)

$\dfrac{l}{w} \ge 5$ ----------(C)

Combining (B) and (C) yields the following relation:

$$5 \le \frac{l}{w} \le t \cdot 10^6 m^{-1}$$
$$5 \le t \cdot 10^6 m^{-1}$$
$$5 \times 10^{-6} m \le t$$

Therefore, from physical constraints alone $t_{\min} = 5 \times 10^{-6} m = 5 \mu m$

Because we are forced to use a beam thickness greater than 2 μm, in the cost function the term that involves the step function is in the "active" state. We therefore find ourselves in a linear regime where any increases in the beam area and/or the beam thickness beyond prescribed physical minima will add to the cost of the part.

From (A), $l = gt / 9.2712 \times 10^{-9}$. The l corresponding with the $t_{\min}$ of 5 μm and the minimum gap (0.5 μm) is given by: $l_{\min} = 0.5 \times 10^{-6} \times 5 \times 10^{-6} / (9.2712 \times 10^{-9}) = 2.696 \times 10^{-4} m \approx 270 \mu m$ **.**

The w follows from (B):

$$\frac{l}{t \cdot 10^6} m \le w$$

$$\frac{270\times10^{-6}}{\left(5\times10^{-6}\right)\cdot 10^6} m = 54\times10^{-6} m \le w$$

The smallest w we can have and still satisfy all of the physical constraints associated with the problem is therefore:

$$w_{\min} = 54\times10^{-6} m = 54\mu m$$

We also know from (6) that the condition $w \ge 10t$ must be satisfied. Since $t_{\min} = 5\times10^{-6} m = 5\mu m$ and $w_{\min} = 54\times10^{-6} m = 54\mu m$, this last condition is met, and we have therefore found our design values.

Thus $\underline{g = 0.5\mu m,\ t = 5\mu m,\ w = 54\mu m,\ l = 270\mu m}$. Substituting into equations (1)-(4) and (7), we get:

V_c= 9.98 V, R = 10Ω, and the minimum cost is: $54\times270\times200\times10^{-6} + (5-2)\times2 = 8.92\$$

(c) Process: (Note that the actual dimensions of the device are slightly larger than those of the cantilever. Hence the cost will be slightly higher than that calculated above.

1. **Start with a silicon wafer, perform RCA clean with HF dip.**
2. **LPCVD 0.2 µm of silicon nitride to act as insulator between the electrode and switch along the substrate path.**
3. **LPCVD 0.2 µm of polysilicon.**
4. **Perform photolithography using positive photoresist (not shown) and Mask 1 to define the electrode.**
5. **Dry-etch the polysilicon using reactive-ion etching. Then ash resist and perform RCA clean (without HF dip).**
6. **PECVD 0.5 µm of sacrificial oxide**
7. **Perform photolithography using positive photoresist (not shown) and Mask 2 to pattern the sacrificial layer.**
8. **Wet-etch the oxide using BOE. Then ash resist and perform RCA clean (without HF dip).**
9. **LPCVD 5 µm of polysilicon.**
10. **Perform photolithography using positive photoresist (not shown) and Mask 3 to define the cantilever.**
11. **Dry-etch the polysilicon using reactive-ion etching. Then ash resist and perform RCA clean (without HF dip).**
12. **Release the cantilever by etching the oxide with BOE followed by super-critical freeze drying.**

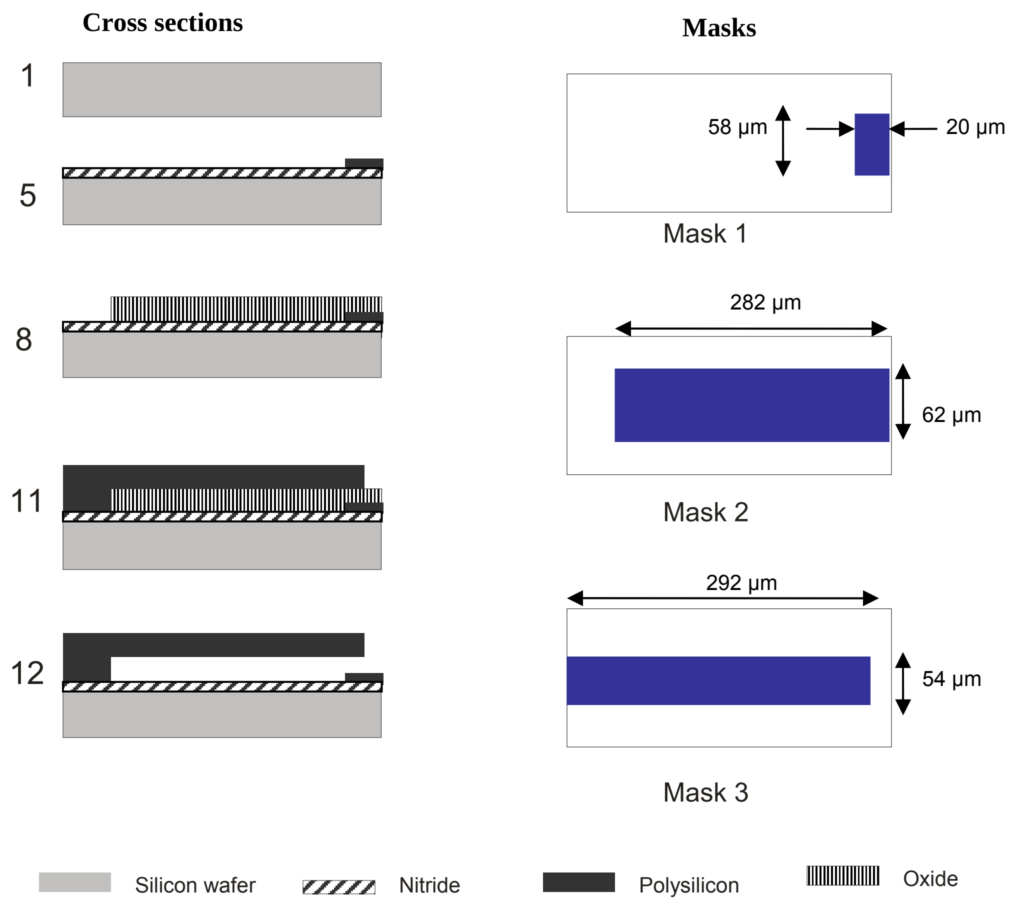


Figure 1. Process flow and mask set for micromachined cantilever switch

Problem 6.11 (7 pts): A simple MEMS resonator

(a) For a doubly clamped cantilever beam, the effective spring constant is:

$$k = \frac{16Ewt^3}{l^3}$$ (See p. 690 in Gere & Timoshenko, Mechanics of Materials, 4th Ed.)

The mass is :
$$m_{actual} = \rho\, lwt$$
$$m_{eff} = 0.4 m_{actual}$$

The pull in voltage is $$V_{PI} = \sqrt{\frac{8kg_o{}^3}{27\varepsilon A_{cap}}} = \sqrt{\frac{8\times 16Ewt^3 g_o{}^3}{27\varepsilon\, l_{elec} wl^3}} = \sqrt{\frac{128Et^3 g_o{}^3}{27\varepsilon\, l_{elec} l^3}} \quad \text{-------- (1)}$$

(b) Based on the linearized model in the text:

$$C_0 = \frac{\varepsilon A_{cap}}{\hat{g}_o} = \frac{\varepsilon\, l_{elec} w}{\hat{g}_o}, \qquad k' = k - \frac{V_o^2 C_o^2}{\varepsilon A_{cap} \hat{g}_o} = k - \frac{V_o^2 \varepsilon\, l_{elec} w}{\hat{g}_o^3}, \qquad \text{and } \phi = \frac{C_o V_o}{\hat{g}_o} = \frac{\varepsilon\, l_{elec} w V_o}{\hat{g}_o{}^2}$$

(c) When we refer the impedances to the electrical side, we multiply each impedance on the mechanical side by $(1/\varphi)^2$, thus:

$$Z_{C_1} = \frac{1}{\phi^2} Z_{1/k'} = \frac{\hat{g}_o^4}{(\varepsilon\, l_{elec} w V_o)^2} \frac{\left(k - \frac{V_o^2 \varepsilon\, l_{elec} w}{\hat{g}_o^{\;3}}\right)}{s} = \frac{1}{C_1 s}$$

$$\Rightarrow C_1 = \frac{\phi^2}{k'} = \frac{(\varepsilon\, l_{elec} w V_o)^2}{k\hat{g}_o^4 - V_o^2 \varepsilon\, l_{elec} w \hat{g}_o}$$

$$Z_L = \frac{Z_m}{\phi^2} = \frac{\rho\, lwt\hat{g}_o^4\, s}{(\varepsilon\, l_{elec} w V_o)^2} = Ls$$

$$\Rightarrow L = \frac{m}{\phi^2} = \frac{\rho\, lwt\hat{g}_o^4}{(\varepsilon\, l_{elec} w V_o)^2}$$

$\hat{g}_o$ is found by solving: $\hat{g}_o = g_o - \dfrac{V_o^2 \varepsilon\, l_{elec} w}{2k\hat{g}_o^2}$

(d) $$Z_{in} = Z_{C_o} // (Z_{C_1} + Z_L) = \frac{1}{sC_o} // \left(\frac{1}{sC_1} + sL\right) = \frac{1 + LC_1 s^2}{(s)(C_o + C_1 + LC_o C_1 s^2)} = \frac{\frac{1}{sC_0}\left(s^2 + \frac{1}{LC_1}\right)}{s^2 + \frac{1}{L}\left(\frac{1}{C_1} + \frac{1}{C_0}\right)} \qquad \text{-------- (2)}$$

This is a third order system with 3 poles and 2 zeros.
Substituting $s = j\omega$ and setting the denominator and numerator to zero to find the poles and zeros respectively, we find that:

The poles occur at s=0 (ω = 0) and s=± jω where $\omega = \sqrt{\dfrac{C_o + C_1}{LC_o C_1}}$

The zeros occur at s=± jω where $\omega = \sqrt{\dfrac{1}{LC_1}}$

(e) When $V_0 = 0$, $\omega_1 \rightarrow \sqrt{\dfrac{k}{\rho\, lwt}} = \dfrac{4t}{l^2}\sqrt{\dfrac{E}{\rho}}$ ---------**(3)**

$$\omega_2 = \sqrt{\frac{1}{LC_1}} = \sqrt{\frac{k - V_o^{\,2} \varepsilon\, l_{elec} w / \hat{g}_o^{\;3}}{\rho\, lwt}} = \sqrt{\frac{16Ewt^3 / l^3 - V_o^{\,2} \varepsilon\, l_{elec} w / \hat{g}_o^{\;3}}{\rho\, lwt}}$$

When $V_0 = \alpha V_{PI}$,

$$= \sqrt{\frac{16Ewt^3 / l^3 - \alpha^2 \times 128 Et^3 g_o^{\;3} \varepsilon\, l_{elec} w / (27 \varepsilon\, l_{elec} l^3 \hat{g}_o^{\;3})}{\rho\, lwt}}$$

$$= \sqrt{\frac{16Et^2 - \alpha^2 \times 128 (g_o / \hat{g}_o)^3 Et^2 / 27}{\rho\, l^4}}$$

$$= \frac{4t}{l^2}\sqrt{\frac{E}{\rho}\left(1 - \frac{8\alpha^2 (g_o / \hat{g}_o)^3}{27}\right)} \qquad \text{-----------}\,\textbf{(4)}$$

$$\hat{g}_o = g_o - \frac{\alpha^2 V_{PI}^2 \varepsilon\, l_{elec} w}{2k\hat{g}_o^2} = g_o\left[1 - \frac{4\alpha^2}{27}\left(\frac{g_o}{\hat{g}_o}\right)^2\right] \text{--------------------}\,\textbf{(5)}$$

(f) As $\alpha \to 1$ in equation (4) above, $V \to V_{PI}$, $\hat{g}_o \to \frac{2}{3} g_o$, and thus: $\omega_2 \to \frac{4t}{l^2}\sqrt{\frac{E}{\rho}(1-\alpha^2)} = \omega_1\sqrt{1-\alpha^2}$

At V_0=0, $\omega_2 = \omega_1$.

At V_0=0.95V_{PI}, using the above approximation: $\omega_2 = \omega_1\sqrt{1-0.95^2} = 0.31\omega_1$

And thus the maximum change in ω_2 according to this approximation is: $\frac{0.31-1}{1} \times 100 = -69\%$

The actual change in ω_2 will be less than that calculated above because the approximate form for ω_2 is really only accurate as $\alpha \to 1$.

In fact, if we solve (5) numerically , we get $\hat{g}_o = 0.7805 g_o$

Substituting in (4) gives: $\omega_2 = \frac{4t}{l^2}\sqrt{\frac{E}{\rho}\left(1 - \frac{8 \times 0.95^2 (1/0.7805)^3}{27}\right)} = \frac{2.646t}{l^2}\sqrt{\frac{E}{\rho}}$ **--------------(6)**

Thus the *actual* maximum change in ω_2 will only be : $\frac{2.646-4}{4} \times 100 = -33.85\%$

(g) From (3), $l = \sqrt{\frac{4t}{\omega_1}\sqrt{\frac{E}{\rho}}} = \sqrt{\frac{4 \times 1 \times 10^{-6}}{2\pi \times 10^6} \times \sqrt{\frac{150 \times 10^9}{2300}}} = 72\mu m$

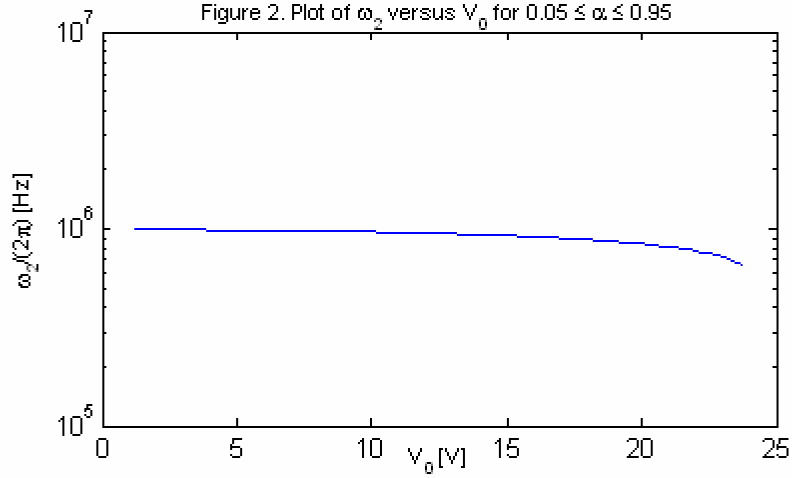
Hence l_{elec} must be ≤ 7.2 μm. Use l_{elec} = 7 μm.

Now from (1): $g_o = \left(\frac{27\varepsilon\, l_{elec} V_{PI}{}^2 l^3}{128 E t^3}\right)^{\frac{1}{3}} = 0.27\mu m$

w must be ≤ l/5 for the structure to act as a beam (rather than a plate), hence we will pick w = 10 μm.

Figure 2 below plots $f_2 = \omega_2 / (2\pi)$ versus V_o as V_o varies from 0.05V_{PI} to 0.95V_{PI} . Figure 3 is a Bode plot of Z_{in} at V_o = 0.05V_{PI} and V_o = 0.95V_{PI} .The MATLAB code is attached at the end of the solution.

Figure 2. Plot of ω_2 versus V_0 for 0.05 ≤ α ≤ 0.95



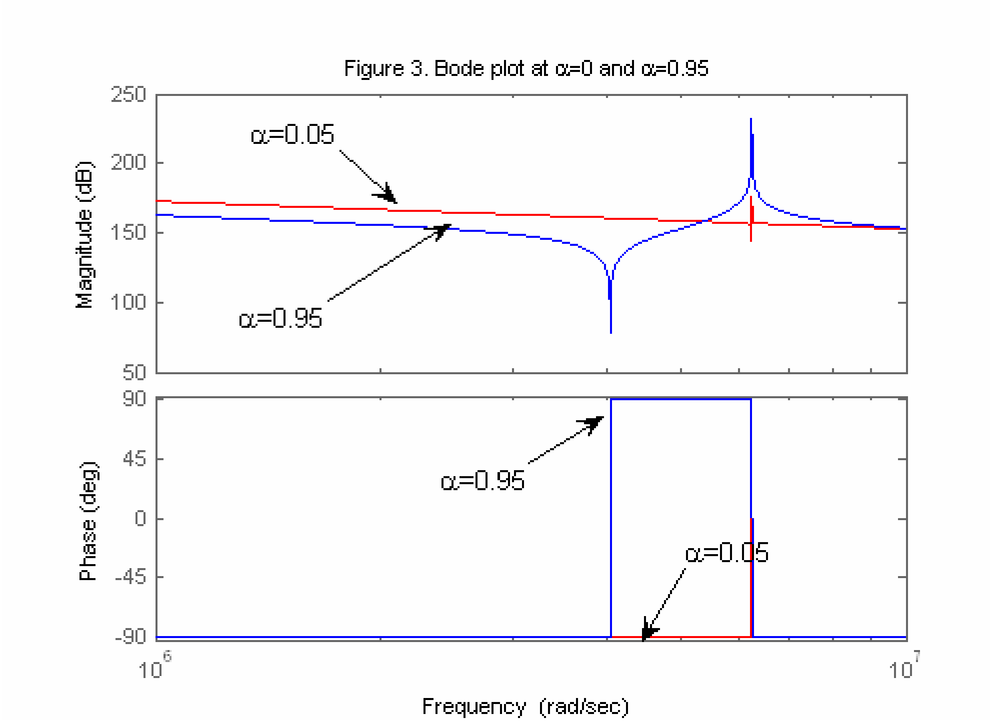


Figure 3. Bode plot at α=0 and α=0.95
Magnitude (dB)
250
200
150
100
50
α=0.05
α=0.95
Phase (deg)
90
45
0
-45
-90
α=0.95
α=0.05
10^6
10^7
Frequency (rad/sec)

```matlab
% MATLAB code for prob 6.11

function out=prob6_11()

g0 = 0.27e-6;
lelec = 7e-6;
w = 10e-6;
l = 71e-6;
t = 1e-6;

e = 8.85e-12;
rho = 2300;
E = 150e9;

A = lelec * w;

k = 16*(E*w*t^3/l^3);
m = t*l*w*rho;

VPI = sqrt(8*k*g0^3/(27*e*A));

V = linspace(0.05*VPI,0.99*VPI,100);

for in=1:length(V)
    V0 = V(in);
    G = fzero(@(G) gap(G,V0,k,e,A,g0),g0);
    C0 = e*A/G;
    phi = C0*V0/G;
    kp(in) = k*(1-C0^2*V0^2/(e*A*k*G));
    C1 = phi^2/kp(in);
    L = m/phi^2;
    w_r(in) = 1/(2*pi)*sqrt(1/(L*C1));
    if in==1
        Z1 = tf([1/C0 0 1/(L*C0*C1)],[1 0 1/L*(1/C0+1/C1) 0]);
    end
    if in==100
        Z2 = tf([1/C0 0 1/(L*C0*C1)],[1 0 1/L*(1/C0+1/C1) 0]);
    end
end


w = logspace(6,7,5000);
bode(Z1,'r',Z2,'b',w)
figure
semilogy(V,w_r)


function y = gap(G,V0,k,e,A,g0)
y = G - g0 + e*A*V0^2/(2*k*G^2);
```