6.777J/2.372J Design and Fabrication of Microelectromechanical Devices
Spring Term 2006

Massachusetts Institute of Technology

PROBLEM SET 6 (10 pts)

Issued: Lecture 14 Due: Lecture 16

Problem 13.9 (3 pts): Calculating fluidic resistances

Here we will examine the fluidic resistances of square versus rectangular channels. Assume a channel with cross-section of 50 μm x 50 μm and length L=1 cm, and that the working fluid is water.

a) Using a parallel-plate Poiseuille flow approximation, calculate the fluidic resistance of the channel in Pa/(μl/min).

b) The parallel-plate flow approximation is not strictly valid when the width of the channel approaches the height. A more general relation between flow and pressure for a rectangular channel of width W and height h (where $W \geq h$), is

$$Q = \frac{W \cdot h^3}{12\eta}\left|\frac{dP}{dx}\right|\left[1 - \frac{192 \cdot h}{\pi^5 \cdot W}\sum_{n=0}^{\infty}\frac{\tanh\left((2 \cdot n+1)\frac{\pi \cdot W}{2 \cdot h}\right)}{(2 \cdot n+1)^5}\right]$$

Use this relation to calculate the resistance of the channel. What is the error of the parallel-plate approximation?

c) Determine analytically or numerically the minimum W/h where the Poiseuille approximation has a 10% deviation from the exact solution.

Problem 13.10 (4 pts): Timescales in microfluidic flows

One commonly assumes that in the creeping flow regime encountered in microfluidics, flows start and stop instantaneously. However, what happens in the real world, where Re is not exactly zero? In this problem we will find out.

a) First, we will derive the relevant dimensionless quantities. Starting from the incompressible Navier-Stokes equation in the text (Eq. 13.25), assume incompressible flow, neglect gravity, and expand out the total derivative. What is the N-S equation now?

b) Next, non-dimensionalize the N-S equation using the following relations:

$$\tilde{u} = U/U_0, \tilde{t} = t/\tau, \tilde{P} = P/\eta U_0 / L, \tilde{\nabla} = L\nabla, \tilde{\nabla}^2 = L^2\nabla^2$$

where U_0, τ, and L are a characteristic velocity, timescale, and length scale for the flow. Place the non-dimensional N-S equation in the form of

$$\mathrm{Re}\left(\frac{1}{Sr}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \cdot \tilde{\nabla}\tilde{u}\right) = -\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2\tilde{u}$$

What are Re and Sr? One now sees how the L in the Reynolds number comes from the characteristic dimension over which spatial change occurs (e.g., from $\tilde{\nabla} = L\nabla$).

c) In creeping flow we can neglect the convective term leading to

$$\frac{\mathrm{Re}}{Sr}\frac{\partial \tilde{u}}{\partial \tilde{t}} = -\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2\tilde{u}$$

If Re<<Sr we return to our quasistatic creeping flow equation where time doesn't matter. Let's examine this coefficient. The Re and Sr numbers can be represented as ratios of timescales. First, express Sr and Re as

$$Sr = \frac{\tau}{\tau_c} \qquad\qquad \mathrm{Re} = \frac{\tau_v}{\tau_c}$$

What are τ_c and τ_v? What do they represent? The ratio is then

$$\frac{\mathrm{Re}}{Sr} = \frac{\tau_v}{\tau}$$

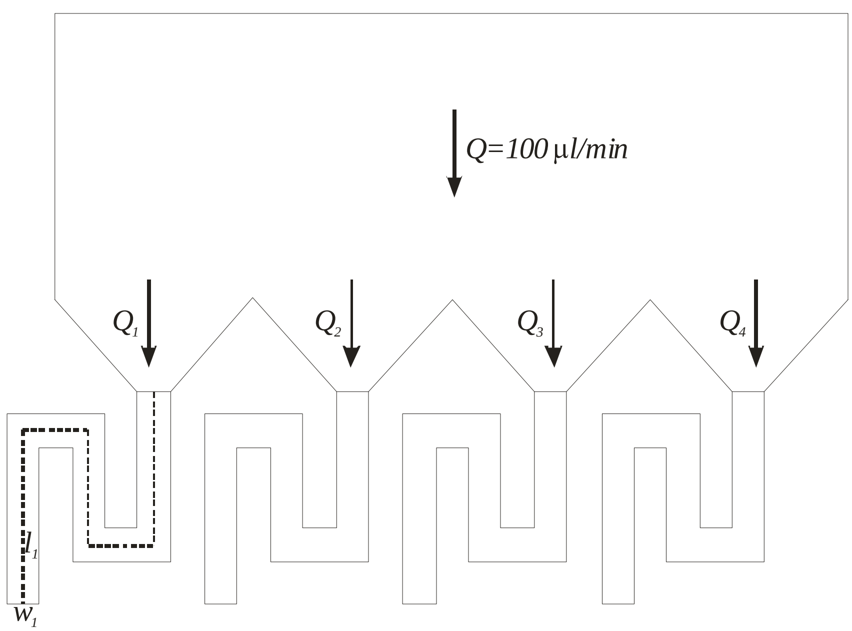
and thus the flow is quasistatic when τ_v is much less than the timescale imposed by the external force.

d) Suppose you have a square 100 μm × 100 μm microchannel filled with water ($\eta = 10^{-3}$ Pa-s, $\rho_m = 1000$ kg/m^3). You apply a step input of pressure. Approximately how long will it take for the flow to reach steady state? Which timescale dominates the flow development in this case?

Problem 13.11 (3 pts): Microfluidic networks and fabrication variations

We commonly create microfluidic networks that take an input flow and distribute that flow into a number of streams. By changing the geometries of the fluidic channels we can create defined fluidic resistors and thus defined flowrate variations across an array. In this problem we will examine how to do this, and, importantly, how robust those predetermined variations are to process non-uniformities.

Given an input volumetric flowrate of water ($\eta = 0.001$ Pa-s at 100 μl/min) to the network shown in the diagram, we would like to establish a 1:3:9:27 ratio in volumetric flowrate across the array (Q1:Q4). The height of the channels is fixed at 50 μm, while the width must be ≥150 μm. The pressure at the channel exits is fixed and the maximum pressure drop across the channels must be ≤1 psi. Finally, we want to minimize the total chip area. Approximating the area of each section n as $2l_nw_n$ (to incorporate spaces between adjacent channels), we must keep the total area of the resistive channels to <25 mm^2. You can assume parallel-plate Poiseuille flow.



a) Determine a set of geometries (l_1, w_1, l_2, w_2, etc.) of the channels that meets the above requirements, and then calculate the flow in each channel and the pressure drop across the channels.

b) Now assume that there is a fabrication variation across the chip, such that the channel height varies 10% across the chip. Assume first that the channel height increases as one goes from channels 1-4, and then that the channel height decreases as one goes from channels 1-4. How does each of these variations change the flowrate ratios across the array? You may assume that the channel height is constant for a given section, and increases stepwise from section to section.