ESTIMATION and CONFIDENCE INTERVALS

Point Estimation: find best values for parameters of a distribution

Should be
- unbiased. Expected value of estimate should be true value
- minimum variance: Should be estimator with smallest variance

Interval Estimation: give bounds that contain actual value with a given probability
- must know sampling distribution!

Case 1: Variance known (e.g. from historical data)
Estimate mean to some interval to $(1-\alpha)100\%$ confidence

$$\bar{x} - Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Case 2: Estimate mean when variance is unknown

$$\bar{x} - t_{\alpha/2, n-1} \cdot \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{\alpha/2, n-1} \cdot \frac{s}{\sqrt{n}}$$

Case 3: Estimate of variance

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2, n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha/2, n-1}}$$

⋮

many other cases (see Montgomery)

Hypothesis Testing

A statistical hypothesis is a statement about the parameters of a probability distribution

$H_0 \triangleq$ Null hypothesis e.g. $H_0: \mu = \mu_0$

$H_1 \triangleq$ Alternative hypothesis $H_1: \mu \neq \mu_0$

In general, we formulate our hypothesis, generate a random sample, compute a statistic, and then seek to reject H_0 or fail to reject (accept) H_0 based on probabilities associated with the statistic.

Error Types

Two types of errors of concern:

$$\alpha = \Pr(\text{Type I error}) = \Pr(\text{reject } H_0 \mid H_0 \text{ is true})$$

$$\beta = \Pr(\text{Type II error}) = \Pr(\text{accept } H_0 \mid H_0 \text{ is false})$$

also $\text{POWER} \triangleq 1-\beta = \Pr(\text{reject } H_0 \mid H_0 \text{ is false})$
i.e. correctly rejecting H_0

Usually specify α, then design sampling procedure so that β is acceptably small (e.g. by choosing sample size).

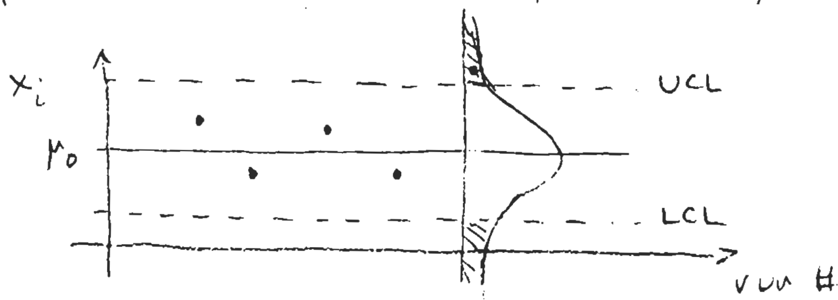
Variable Control Chart

When a quality characteristic is espressed in terms of a continuous numerical measure (e.g. thicknesses, film stress, transistor saturation current), the variable chart methods are used.

Generally, one typically wants to control both the MEAN and VARIANCE of the characteristic. First, we consider an even simpler case - where we only take one random sample or measurement:

Relationship to Hypothesis Testing

Suppose we know historically that $x \sim N(\mu_0, \sigma_0^2)$.



We can consider each sample x_i as a test

$$
\begin{aligned}
H_0: &\quad x_i = \mu_0 \\
H_1: &\quad x_i \neq \mu_0
\end{aligned}
$$

How do we set the Control Limits?

- Decide α risk we're willing to live with:

$\Rightarrow$ What Probability of a <u>false alarm</u> is acceptable?

- Example: $\alpha = 0.001$ $\sim$ Can abso[illegible] [illegible]1000 data points outside UCL/LCL generating an alarm when process is really in control

SET CL at $\mu_0 \pm z_{\alpha/2} \cdot \sigma_0$

$$= \mu_0 \pm z_{0.0005} \cdot \sigma_0$$

$$= \mu_0 \pm 3.29\ \sigma_0$$

- Often, accept α-risk on each side of chart, i.e. 1-sided α-risk of 0.001; or $\alpha = 0.002$ level of significance

SET CL at $\mu_0 \pm z_{\alpha} \cdot \sigma_0$

$$= \mu_0 \pm 3.09\ \sigma_0$$

← Very close to the typical (US) "3 sigma" limits.

<u>Additional Rules</u>

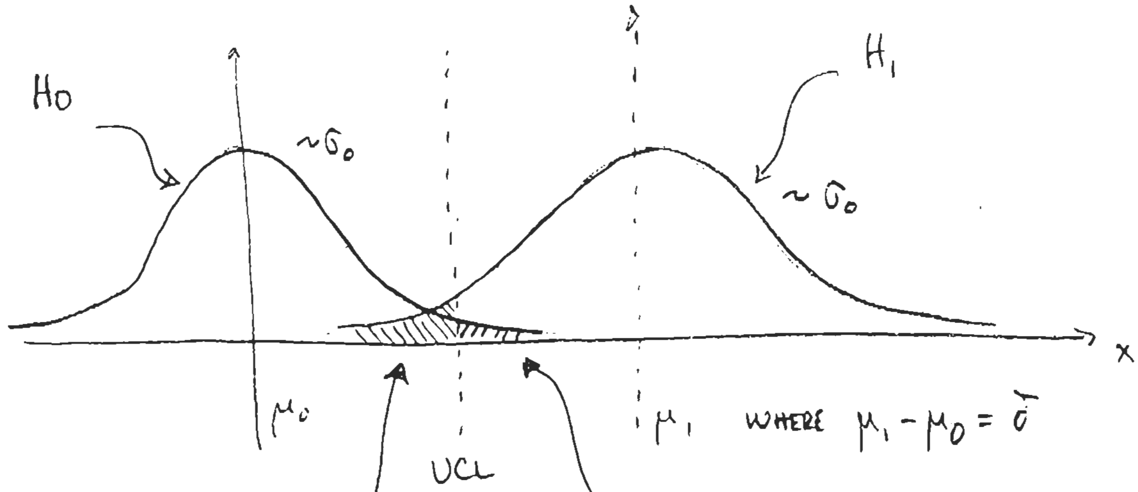
The goal of the chart is to help detect and diagnosis "out of control", i.e. NON-random, behavior. Other patterns can also be considered:

Caution:

$$\alpha = 1 - \prod_i (1 - \alpha_i)$$

OVERALL false alarm rate increases!

α vs. β RISK



WHERE $\mu_1 - \mu_0 = \delta$

$\frac{\alpha}{2}$ risk: reject $H_0: x_i = \mu_0 \mid \mu = \mu_0$

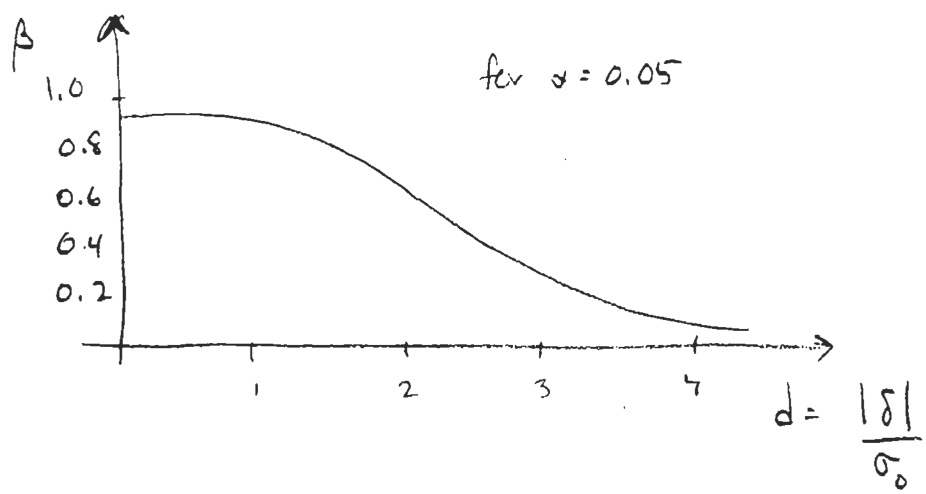
$\frac{\beta}{2}$ risk: accept $H_0 \mid H_1$

On the standard normal curves:

$$\beta = \Phi\left(z_{\alpha/2} - \frac{\delta}{\sigma_0}\right) - \Phi\left(-z_{\alpha/2} - \frac{\delta}{\sigma_0}\right)$$

So β depends on the deviation we hope to detect...

e.g



OPERATING CHARACTERISTIC CURVE

(Similar to ROC in signal detection)

Sampling and Control Charts

- A key limitation in measuring one individual is that we have NO estimate for the variance.

  $\Rightarrow$ important to monitor both mean and variance

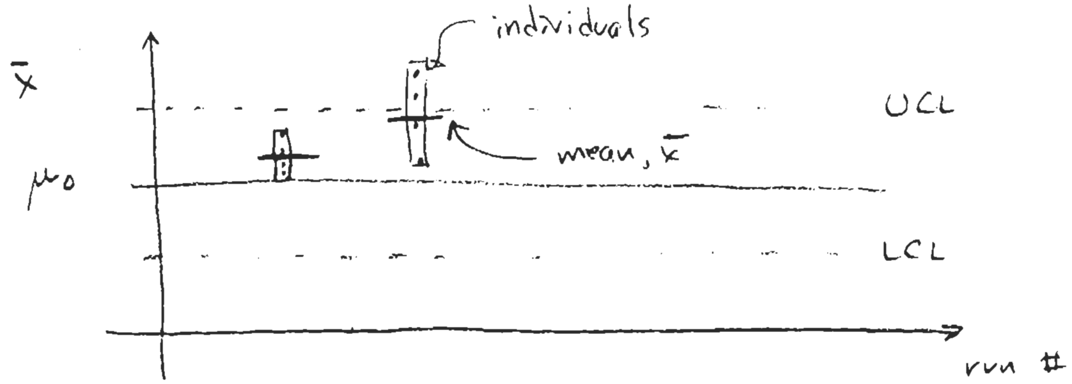
- Another key limitation: in the single variable case, we ASSUMED a distribution $N(\mu_0, \sigma_0^2)$

$\bar{x}$ Chart

Suppose we randomly sample n individuals. Then by Central Limit Theorem

$$\bar{x} \sim N(\mu_w, \sigma_w^2) = N\left(\mu_0, \frac{\sigma_0^2}{n}\right)$$

regardless of distribution of x_i. If we take "large enough" samples, we are robust to non-normality in our underlying process! (samples of 4-5 usually sufficient)



$H_0: \bar{x}_i = \mu_0$ BUT now $\bar{x} \sim N\left(\mu_0, \frac{\sigma_0^2}{n}\right)$

$H_1: \bar{x}_i \neq \mu_0$ so

SET C_L at $\mu_0 \pm Z_{\alpha/2} \cdot \sigma_w = \mu_0 \pm Z_{\alpha/2} \cdot \frac{\sigma_0}{n}$

β- risk in $\bar{x}$ charts

When using $\bar{x}$ charts formed by sampling, now the "false accept" risk β depends on sample size:

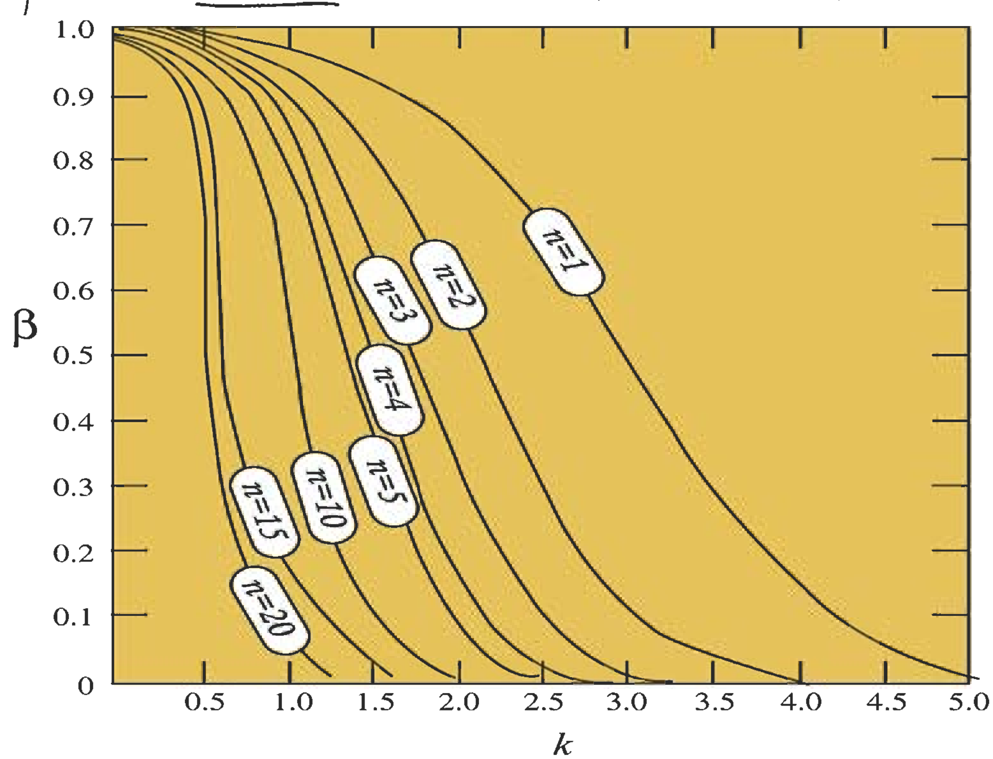
$$\beta \triangleq \Pr\left(LCL \le \bar{x} \le UCL \;\middle|\; \mu = \mu_1 = \mu_0 + k\hat{\sigma}_0 \right)$$

Say we use $\mu_0 \pm 3\sigma_0/\sqrt{n}$ as control limits. Then

$$\beta = \Phi\left[\frac{UCL - (\mu_0 + k\hat{\sigma}_0)}{\hat{\sigma}_0/\sqrt{n}} \right] - \Phi\left[\frac{LCL - (\mu_0 + k\hat{\sigma}_0)}{\hat{\sigma}_0/\sqrt{n}} \right]$$

$$= \Phi\left[3 - k\sqrt{n} \right] - \Phi\left[-3 - k\sqrt{n} \right]$$

$\therefore$ For a given α-risk (e.g. that of $3\hat{\sigma}_0/\sqrt{n}$ limits), the β-risk <u>decreases</u> as we increase n:



Operating-characteristic curves for the $\bar{x}$ chart with 3-sigma limits. $\beta = P$(not detecting a shift of $k\sigma$ in the mean on the first sample following the shift).