Massachusetts Institute of Technology

Department of Computer Science and Electrical Engineering

6.801/6.866 Machine Vision Quiz 1 Solutions

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PROBLEM 1 (15 points)

Part a (5 points).

From the given statement, we have

$$E_1(x,y) = \rho(x,y) * R_1(p,q)$$
$$E_2(x,y) = \rho(x,y) * R_2(p,q)$$

Thus, if we define

$$E'(x,y) = \frac{E_1(x,y)}{E_2(x,y)}$$
$$R'(p,q) = \frac{R_1(p,q)}{R_2(p,q)}$$

we have

$$E'(x,y) = R'(p,q)$$

which is independent of albedo $\rho(x,y)$.

Part b (5 points).

Assume the direction for light source is $\mathbf{S_1} = (-\mathbf{p_1}, -\mathbf{q_1}, \mathbf{1})^{\mathbf{T}}$ and $\mathbf{S_2} = (-\mathbf{p_2}, -\mathbf{q_2}, \mathbf{1})^{\mathbf{T}}$. For lambertian surface, we have

$$R_1(p,q) = \frac{1 + p_1 p + q_1 q}{\sqrt{1+p^2+q^2}\sqrt{1+p_1^2+q_1^2}}$$

$$R_2(p,q) = \frac{1 + p_2 p + q_2 q}{\sqrt{1+p^2+q^2}\sqrt{1+p_2^2+q_2^2}}$$

Thus,

$$\begin{aligned} R'(p,q) &= \frac{R_1(p,q)}{R_2(p,q)} \\ &= \frac{1+p_1p+q_1q}{1+p_2p+q_2q}\frac{\sqrt{1+p_2^2+q_2^2}}{\sqrt{1+p_1^2+q_1^2}} \end{aligned}$$

If $R'(p,q)$ is constant, we know that

$$\frac{1+p_1p+q_1q}{1+p_2p+q_2q} = k > 0$$

has to be held.

i.e.,

$$(1+p_1p+q_1q) - k(1+p_2p+q_2q) = 1 - k + (p_1 - kp_2)p + (q_1 - kq_2)q = 0$$

Therefore, we know that isophotes for newly defined $E'(x,y) = R'(p,q)$ are straight lines for synthesized image even if the surface is Lambertian.

The straight lines are parallel if $(p_1 - kp_2) : (q_1 - kq_2)$ is constant, which requires

$$p_1 : q_1 = p_2 : q_2$$

Note: 1. The expression should allow the case when one variable (p_1, q_1, p_2, q_2) is equal to zero. 2. The light sources are not necessarily in parallel. But their project in (p,q) plan is in the same direction.

Part c (5 points).

The family of straight lines in p and q

$$(1+p_1p+q_1q) - k(1+p_2p+q_2q) = 0 \tag{1}$$

will pass a point which satisfies the following equations.

$$1 + p_1 p_0 + q_1 q_0 = 0$$
$$1 + p_2 p_0 + q_2 q_0 = 0$$

Thus,

$$\begin{aligned}
\begin{bmatrix} p_0 \\ q_0 \end{bmatrix} &= \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\
&= \frac{1}{p_1 q_2 - p_2 q_1} \begin{bmatrix} q_1 - q_2 \\ p_2 - p_1 \end{bmatrix}
\end{aligned}$$

Method 2:
We can also rewrie equation 1 in the format as
$q - q_0 = c(p - p_0)$
which shows that the line cluster must pass the point (p_0, q_0).
From equation 1, we know that:

$$q = \frac{-p_1 + kp_2}{q_1 - kq_2} p + \frac{k - 1}{q_1 - kq_2}$$

Letting $c = (-p_1 + kp_2)/q_1 - kq_2$, equation 2 becomes

$$\begin{aligned}
q &= cp + \frac{c(q_2 - q_1) + (p_2 - p_1)}{-q_1 p_2 + p_1 q_2} \\
q - \frac{(p_2 - p_1)}{-q_1 p_2 + p_1 q_2} &= c(p - \frac{c(q_1 - q_2) + (p_2 - p_1)}{-q_1 p_2 + p_1 q_2}
\end{aligned}$$

which is what we need.
Method 3:
If people just calculate the intersection of two lines:

$$(1 + p_1 p + q_1 q) - k_1(1 + p_2 p + q_2 q) = 0$$
$$(1 + p_1 p + q_1 q) - k_2(1 + p_2 p + q_2 q) = 0$$

The solution they obtain should be independent of variable parameter k_1 and k_2, which proves that the family of lines pass though common points.

Actually, this common point in gradient space will give the highest accuracy in recovering surface orientation. It is because surface orientation (p, q) at this point is independent of "brightness", i.e., the estimation is insensitive to the measurement error of image brightness. (Intuitively, isophotes (in gradient space) are closest to each other at this point.)

People can obtain the conclusion by calculating the sensitivity of "brightness" to surface orientation as follows.

$$\begin{aligned}
E'(x,y) &= R'(p,q) = \alpha\frac{1+p_1p+q_1q}{1+p_2p+q_2q} \\
\frac{\partial R'(p,q)}{\partial p} &= \alpha\frac{p_1}{1+p_2p+q_2q} - \alpha p_2\frac{1+p_1p+q_1q}{(1+p_2p+q_2q)^2} \\
&= \alpha\frac{p_1(1+p_2p+q_2q)-p_2(1+p_1p+q_1q)}{(1+p_2p+q_2q)^2} \\
&= \alpha\frac{p_1-p_2+q(p_1q_2-p_2q_1)}{(1+p_2p+q_2q)^2}
\end{aligned}$$

Obviously, when $1 + p_2p + q_2q = 0$, there will be high estimation accuracy in recovering surface orientation p.

Similarly, if we define

$$\begin{aligned}
E'(x,y) &= R'(p,q) = \alpha\frac{1+p_2p+q_2q}{1+p_1p+q_1q} \\
\frac{\partial R'(p,q)}{\partial q} &= \alpha\frac{q_1-q_2+p(q_1p_2-q_2p_1)}{(1+p_1p+q_1q)^2}
\end{aligned}$$

when $1 + p_1p + q_1q = 0$, there will be high estimation accuracy in recovering surface orientation q.

(Note: We can reach the same conclusion when we calculate the relative accuracy which is:

$$\frac{1}{R'(p,q)}\frac{\partial R'(p,q)}{\partial q} = \alpha\frac{p_1-p_2+q(p_1q_2-p_2q_1)}{(1+p_1p+q_1q)(1+p_2p+q_2q)}$$

)

In sum, around the point which satisfies both

$$1 + p_1 p_0 + q_1 q_0 = 0$$
$$1 + p_2 p_0 + q_2 q_0 = 0$$

the accuracy of orientation estimation is high.

We notice that the orientation corresponding to the common points of the isophotos in gradient space is perpendicular to the direction of two original light sources $\mathbf{S_1} = (-\mathbf{p_1}, -\mathbf{q_1}, \mathbf{1})^{\mathbf{T}}$ and $\mathbf{S_2} = (-\mathbf{p_2}, -\mathbf{q_2}, \mathbf{1})^{\mathbf{T}}$.

In sum, two lighting sources should be contained in a plane that is tangent to the surface to our interest in order to receive high estimation accuracy. Besides, we typically set two lighting sources to be perpendicular to each other since $p_1 q_2 - p_2 q_1$ should be large enough to garantee high accuracy, i.e., $\mathbf{S_1} = (-\mathbf{p_1}, -\mathbf{q_1}, \mathbf{1})^{\mathbf{T}}$ and $\mathbf{S_2} = (-\mathbf{p_2}, -\mathbf{q_2}, \mathbf{1})^{\mathbf{T}}$ should be far away from being parallell.

Note: At point (p_0, q_0), $R_1(p_0, q_0) = R_2(p_0, q_0) = 0$, thus $R'(p_0, q_0)$ could be defined arbitrarily. That is also the reason this point is the intersection of multiple isophotes.

PROBLEM 2 (25 points)

Part a (4 points).

The derivative of

$$M(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$$

are

$$\begin{aligned} M_x(x, y) &= 2ax + by + d \\ M_y(x, y) &= bx + 2cy + e \\ M_{xx}(x, y) &= 2a \\ M_{xy}(x, y) &= b \\ M_{yy}(x, y) &= 2c \end{aligned}$$

Evaluated at $(x, y) = (0, 0)$,

$$
\begin{aligned}
M(0,0) &= f \\
M_x(0,0) &= d \\
M_y(0,0) &= e \\
M_{xx}(0,0) &= 2a \\
M_{xy}(0,0) &= b \\
M_{yy}(0,0) &= 2c
\end{aligned}
$$

Thus,

$$
\begin{aligned}
a &= \frac{M_{xx}(0,0)}{2} \\
b &= M_{xy}(0,0) \\
c &= \frac{M_{yy}(0,0)}{2} \\
d &= M_x(0,0) \\
e &= M_y(0,0) \\
f &= M(0,0)
\end{aligned}
$$

Part b (4 points).

From Tyler expansion, we know that

$$
\begin{aligned}
M_x(0,y) &\approx \frac{1}{2}\left(\frac{1}{\varepsilon}\Big(M(\varepsilon,y) - M(0,y)\Big) + \frac{1}{\varepsilon}\Big(M(0,y) - M(-\varepsilon,y)\Big)\right) \\
&= \frac{1}{2\varepsilon}\Big(M(\varepsilon,y) - M(-\varepsilon,y)\Big) \\
M_{xx}(0,y) &\approx \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon}\Big(M(\varepsilon,y) - M(0,y)\Big) - \frac{1}{\varepsilon}\Big(M(0,y) - M(-\varepsilon,y)\Big)\right) \\
&\approx \frac{1}{\varepsilon^2}\Big(M(\varepsilon,y) - 2M(0,y) + M(-\varepsilon,y)\Big) \\
M_{xy}(0,0) &\approx \frac{1}{2\varepsilon}\left(\frac{1}{2\varepsilon}\Big(M(\varepsilon,\varepsilon) - M(-\varepsilon,\varepsilon)\Big) - \frac{1}{2\varepsilon}\Big(M(\varepsilon,-\varepsilon) - M(-\varepsilon,-\varepsilon)\Big)\right)
\end{aligned}
$$

$$= \frac{1}{4\varepsilon^2}\Big(M(\varepsilon,\varepsilon) - M(-\varepsilon,\varepsilon) - M(\varepsilon,-\varepsilon) + M(-\varepsilon,-\varepsilon)\Big)$$

Thus,

$$d = M_x(0,0) \approx \frac{1}{6\varepsilon}\Big(M(\varepsilon,-\varepsilon) - M(\varepsilon,-\varepsilon) + M(\varepsilon,0) - M(\varepsilon,0) + M(\varepsilon,\varepsilon) - M(\varepsilon,\varepsilon)\Big)$$

corresponds to stencil 5.

Similarly, e $(M_y(0,0))$ will corresponds to stencil 3

$$\begin{aligned} a = \frac{M_{xx}(0,0)}{2} \approx \frac{1}{6\varepsilon^2} \Bigg(& \Big(M(\varepsilon,-\varepsilon) - 2M(0,-\varepsilon) + M(-\varepsilon,-\varepsilon)\Big) \\ + & \Big(M(\varepsilon,0) - 2M(0,0) + M(-\varepsilon,0)\Big) \\ + & \Big(M(\varepsilon,\varepsilon) - 2M(0,\varepsilon) + M(-\varepsilon,\varepsilon)\Big)\Bigg) \end{aligned}$$

corresponds to stencil 6.

Similarly, c ($M_{yy}(0,0)/2$) will corresponds to stencil 1.

In sum,

Stencil (1) can be used to estimated c [$M_{yy}(0,0)/2$].

Stencil (2) can be used to estimated b [$M_xy(0,0)$].

Stencil (3) can be used to estimated e [$M_y(0,0)$].

Stencil (4) can be used to estimated f [$M(0,0)$] based on deduction in class.

Stencil (5) can be used to estimated d [$M_x(0,0)$].

Stencil (6) can be used to estimated a [$M_{yy}(0,0)/2$].

Part c (3 points).

From part (b),

$$\begin{aligned} k_1 &= \frac{1}{6\varepsilon^2} \\ k_2 &= \frac{1}{4\varepsilon^2} \\ k_3 &= \frac{1}{6\varepsilon} \end{aligned}$$

$$
\begin{aligned}
k_4 &= \frac{1}{9} \\
k_5 &= \frac{1}{6\varepsilon} \\
k_6 &= \frac{1}{6\varepsilon^2}
\end{aligned}
$$

Part d (5 points).

All points along the direction of gradient can be represented as:

$$
\begin{aligned}
x &= \rho \frac{E_x}{\sqrt{E_x^2 + E_y^2}} \\
y &= \rho \frac{E_y}{\sqrt{E_x^2 + E_y^2}}
\end{aligned}
$$

We notice that the corresponding displacement distance from (x, y) to origin $(0, 0)$ is equal to $|\rho|$.

Substitude into $M(x, y)$,

$$
\begin{aligned}
M(\rho) &= \frac{aE_x^2 + bE_xE_y + cE_y^2}{E_x^2 + E_y^2}\rho^2 + \frac{dE_x + eE_y}{\sqrt{E_x^2 + E_y^2}}\rho + f \\
&= \frac{\frac{1}{2}M_{xx}E_x^2 + M_{xy}E_xE_y + \frac{1}{2}M_{yy}E_y^2}{E_x^2 + E_y^2}\rho^2 + \frac{M_xE_x + M_yE_y}{\sqrt{E_x^2 + E_y^2}}\rho + M
\end{aligned}
$$

The magnitude of gradient reaches maximum when we let

$$
\begin{aligned}
\frac{dM(\rho)}{d\rho} &= \rho\frac{M_{xx}E_x^2 + 2M_{xy}E_xE_y + M_{yy}E_y^2}{E_x^2 + E_y^2} + \frac{M_xE_x + M_yE_y}{\sqrt{E_x^2 + E_y^2}} = 0 \\
\frac{d^2M(\rho)}{d\rho^2} &= \frac{M_{xx}E_x^2 + 2M_{xy}E_xE_y + M_{yy}E_y^2}{E_x^2 + E_y^2} < 0
\end{aligned}
$$

i.e. when

$$\rho = -\frac{M_x E_x + M_y E_y}{M_{xx}E_x^2 + 2M_{xy}E_xE_y + M_{yy}E_y^2}\sqrt{E_x^2 + E_y^2}$$
$$M_{xx}E_x^2 + 2M_{xy}E_xE_y + M_{yy}E_y^2 < 0$$

Thus, the corresponding displacement distance from origin $(0,0)$ to the maximum in brightness gradient in the direction of the local brightness gradient is obtained as above.

To be accurate, the distance is supposed to be $|\rho|$.

Part e (5 points).

From part (d), to garantee the brightness gradient reach maximum instead of minimum, we must have

$$M_{xx}E_x^2 + 2M_{xy}E_xE_y + M_{yy}E_y^2 < 0$$

Part f (4 points).

We define

$$\begin{aligned} R &= M_{xx}E_x^2 + 2M_{xy}E_xE_y + M_{yy}E_y^2 \\ S &= M_xE_x + M_yE_y \\ T &= \sqrt{E_x^2 + E_y^2} \end{aligned}$$

Thus,

$$M(\rho) = \rho^2\frac{R}{2T^2} + \rho\frac{S}{T} + M = 0$$

We know that $M(\rho) = 0$ when

$$\rho_1 = \frac{-S - \sqrt{S^2 - 2RM}}{R} T$$
$$\rho_2 = \frac{-S + \sqrt{S^2 - 2RM}}{R} T$$

When condition in part (e) is met, i.e., $R < 0$, we know that $M(\rho)$ will be negative when

$$\rho > \max(\rho_1, \rho_2) \quad \rho < \min(\rho_1, \rho_2)$$

which violate our assumption that magnitude is non-negative. Therefore, the distance is limited by $\max(|\rho_1|, |\rho_2|)$.

Besides, since $M(x, y)$ is for subpixel approximation and the pixel of interest has survived non-maximum suppression, ρ usually should be bounded by $\sqrt{2}\varepsilon/2$. (Assume that the pixel spacing is ε.)

Proof: Because the pixel at P has survived non-maximum suppresion, we have $M(0) > M(\alpha)$ and $M(0) > M(-\alpha)$ where α is the distance from the pixel to the neighbor pixel in the gradient direction (E_x, E_y). We can see that $\alpha < \sqrt{2}\varepsilon$. ($\alpha$ is at most the diagonal of a square of side ε.) $M(0) > M(\alpha)$ and $M(0) > M(-\alpha)$ imply that

$$M > \alpha^2 \frac{R}{2T^2} + \alpha \frac{S}{T} + M = 0$$
$$M > \alpha^2 \frac{R}{2T^2} - \alpha \frac{S}{T} + M = 0$$

Thus,

$$-\frac{R}{2T^2}\alpha^2 > \pm \frac{S}{T}$$

R is negative. Thus

$$|\frac{R}{2T^2}|\alpha > |\frac{S}{T}|$$
$$\alpha > \frac{|\frac{S}{T}|}{|\frac{R}{2T^2}|}$$

But we know that $\alpha \leq \sqrt{2}\varepsilon$
Hence,

$$\frac{|\frac{S}{T}|}{|\frac{R}{2T^2}|} < \sqrt{2}\varepsilon$$

In sum,

$$|\rho| = \frac{|\frac{S}{T}}{2|\frac{R}{2T^2}|} < \frac{\sqrt{2}\varepsilon}{2}$$

PROBLEM 3a (10 points)

Part a (4 points).

$R(p,q)$ has a unique isolated maximum at $(p,q) = (0,0)$, which corresponds to singular point $(0,0)$. (The assumption does not loose generality since $z(x,y)$ is in the power series of the form. If the singular point is not at origin, we can translate the coordination system.) Therefore, at the singular point $(0,0)$,

$$z_x = s_x + 2ax + 2by = 0$$
$$z_y = s_y + 2bx + 2cy = 0$$

leads to

$$s_x = 0, \; s_y = 0.$$

Part b (3 points).

Because $p = z_x, \; q = z_y, \; z_x = s_x + 2ax + 2by, \; z_y = s_y + 2bx + 2cy$, we have

$$E(x,y) = R(p,q) = p^2 + q^2 = (s_x + 2ax + 2by)^2 + (s_y + 2bx + 2cy)^2$$
$$E_x = 4[(2a^2 + 2b^2)x + (2ab + 2bc)y + as_x + bs_y]$$

$$E_y = 4[(2b^2 + 2c^2)y + (2ab + 2bc)x + bs_x + cs_y]$$
$$E_{xx} = 8(a^2 + b^2),\ E_{yy} = 8(b^2 + c^2),\ E_{xy} = 8(ab + bc)$$
$$E_{xx} - 2E_{xy} + E_{xy} = 8((a - b)^2 + (b - c)^2)$$

It shows that $k = 8$.

Part c (3 points).

If $E_{xx} = E_{yy} = E_{xy} = 8$, then

$$a^2 + b^2 = 1$$
$$b^2 + c^2 = 1$$
$$ab + bc = 1$$

Finally we find out that

$$a = b = c = \pm\frac{1}{\sqrt{2}}$$

There is no unique answer.

Verify:

$$\begin{aligned} b(E_{xx} - E_{yy}) &= \pm\frac{1}{\sqrt{2}}(8 - 8) = 0 \\ (a - c)E_{xy} &= (0)8 = 0 \end{aligned}$$

Thus, we verified that

$$b(E_{xx} - E_{yy}) = (a - c)E_{xy}.$$

Of course we can verify it directly as follows.

$$\begin{aligned} b(E_{xx} - E_{yy}) &= b(8(a^2 + b^2) - 8(b^2 + c^2)) \\ &= (a - c)(8ab + bc) \\ &= (a - c)E_{xy} \end{aligned}$$

PROBLEM 3b (10 points)

Considering the characteristic strip equations:

$$\frac{dx}{d\varepsilon} = R_p \quad \frac{dy}{d\varepsilon} = R_q \quad \frac{dz}{d\varepsilon} = pR_p + qR_q \quad \frac{dp}{d\varepsilon} = E_x \quad \frac{dq}{d\varepsilon} = E_y$$

Thus we have:

$$\begin{aligned}
\frac{d}{d\varepsilon}(E(x,y) - R(p,q)) &= (E_x \frac{dx}{d\varepsilon} + E_y \frac{dy}{d\varepsilon}) - (R_p \frac{dp}{d\varepsilon} + R_q \frac{dq}{d\varepsilon}) \\
&= (E_x R_p + E_y R_q) - (R_p E_x + R_q E_y) \\
&= 0
\end{aligned}$$

$E(x,y) - R(p,q)$ is constant along the characteristic strip. If we know that $E(x,y) = R(p,q)$ at the beginning of the strip, then the equation holds all along the stripe.

PROBLEM 4 (20 points)

Part a (4 points).

Since

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}, \ \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r},$$

we have,

$$\begin{aligned}
E_x &= f^{'}(r)\frac{\partial r}{\partial x} = f^{'}(r)\frac{x}{r}, \qquad E_y = f^{'}(r)\frac{\partial r}{\partial y} = f^{'}(r)\frac{y}{r}, \\
E_{xx} &= f^{''}(r)\frac{\partial r}{\partial x}\frac{x}{r} + f^{'}(r)(\frac{r - x\frac{\partial r}{\partial x}}{r^2}) = f^{''}(r)(\frac{x}{r})^2 + f^{'}(r)(\frac{r - x\frac{x}{r}}{r^2}) = f^{''}(r)(\frac{x}{r})^2 + \frac{f^{'}(r)}{r}\frac{y^2}{r^2} \\
E_{xy} &= f^{''}(r)\frac{\partial r}{\partial y}\frac{x}{r} + f^{'}(r)(\frac{-x\frac{\partial r}{\partial y}}{r^2}) = f^{''}(r)\frac{xy}{r^2} - \frac{f^{'}(r)}{r}\frac{xy}{r^2} \\
E_{yy} &= f^{''}(r)\frac{\partial r}{\partial y}\frac{y}{r} + f^{'}(r)(\frac{r - y\frac{\partial r}{\partial y}}{r^2}) = f^{''}(r)(\frac{y}{r})^2 + f^{'}(r)(\frac{r - y\frac{y}{r}}{r^2}) = f^{''}(r)(\frac{y}{r})^2 + \frac{f^{'}(r)}{r}\frac{y^2}{r^2}
\end{aligned}$$

Part b (4 points).

$$(E_x, E_y)\begin{pmatrix} E_x \\ E_y \end{pmatrix} = E_x^2 + E_y^2 = (f^{'}(r))^2\frac{x^2+y^2}{r^2} = (f^{'}(r))^2$$

$$\begin{aligned}
&(E_y, -E_x)\begin{pmatrix} E_{xx} & E_{xy} \\ E_{xy} & E_{yy} \end{pmatrix}\begin{pmatrix} E_y \\ -E_x \end{pmatrix} = E_x^2 E_{yy} + E_y^2 E_{xx} - 2E_{xy}E_xE_y \\
= \ &(f^{'}(r)\frac{x}{r})^2(f^{''}(r)(\frac{y}{r})^2 + \frac{f^{'}(r)}{r}\frac{y^2}{r^2}) + (f^{'}(r)\frac{y}{r})^2(f^{''}(r)(\frac{x}{r})^2 + \frac{f^{'}(r)}{r}\frac{y^2}{r^2}) \\
&-2(f^{''}(r)\frac{xy}{r^2} - \frac{f^{'}(r)}{r}\frac{xy}{r^2})(f^{'}(r))^2\frac{xy}{r^2} \\
= \ &(f^{'}(r))^3\frac{(x^2+y^2)^2}{r^5} = \frac{(f^{'}(r))^3}{r}
\end{aligned}$$

Thus,

$$\begin{aligned}
\int\int \frac{E_x^2 E_{yy} + E_y^2 E_{xx} - 2E_{xy}E_xE_y}{E_x^2 + E_y^2} dxdy &= \int\int \frac{(f^{'}(r))^3}{r(f^{'}(r))^2} dxdy = \int\int \frac{(f^{'}(r))^3}{r(f^{'}(r))^2} r dr d\theta \\
&= 2\pi \int (f^{'}(r))dr = 2\pi f(r)|_0^{2R} = -2\pi.
\end{aligned}$$

Part c (4 points).

For dark disk in a bright background, the result will be 2π. If the bright circular has an internal dark circular disk-shaped depression, the result will be 0. If there is more than one bright circular disk, the result will be $-2n\pi$, where n is the number of bright circular disk.

Part d (4 points).

The result calculates the Euler number of binary images times -2π. We can think it in this way: 1) It does not change our result if (c) is not strictly circular. We can approximaate a irregular curve with a sequence of short circular arcs. 2) It does not change our result if we translate the circular. 3) It does not change our result if we divide the image into several pieces. 4) From (d), we know that the result is -2π for each solid bright object, 2π for each solid dark object(hole), 0 for each object with only one hole.

Part e (4 points).

Directional 2nd derivative is defined as

$$\frac{v^T H v}{\|v\|^2}$$

where H is the Hessian and v is a vector in the desired direction.

Here below we try to deduct the equation so that we can understand what is the physical meaning behind the definition.

Consider rotating coordinate system angle θ:

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}$$

i.e.,

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}$$

According to chain rules, we have:

$$\begin{aligned} E_{x'} &= E_x\frac{\partial x}{\partial x'} + E_y\frac{\partial y}{\partial x'} = E_x\cos\theta + E_y\sin\theta \\ E_{x'x'} &= E_{x'x}\frac{\partial x}{\partial x'} + E_{x'y}\frac{\partial y}{\partial x'} = E_{xx}(\cos\theta)^2 + 2E_{yx}\sin\theta\cos\theta + E_{yy}(\sin\theta)^2 \end{aligned}$$

Thus, we can see that directional second derivative is also the second derivative along newly defined coordinate system.

If we let

$$\begin{aligned} \cos\theta &= \tfrac{\mp E_y}{\sqrt{E_x^2+E_y^2}} \\ \sin\theta &= \tfrac{\pm E_x}{\sqrt{E_x^2+E_y^2}} \end{aligned}$$

or

$$\begin{aligned} \cos\theta &= \tfrac{\mp y}{\sqrt{x^2+y^2}} \\ \sin\theta &= \tfrac{\pm x}{\sqrt{x^2+y^2}} \end{aligned}$$

we have,

$$E_{x'x'} = \frac{E_{xx}E_y^2 - 2E_{xy}E_xE_y + E_{yy}E_x^2}{E_x^2 + E_y^2},$$

which is exactly the integrand in the integral given.

We notice that the direction is perpendicular to the direction of brightness gradient, i.e., it is tangent to isophotes. Specifically, for this example, the direction of 1st derivative is perpendicular to the line that connect the point and center.

PROBLEM 5a (10 points)

Part a (3 points).

Whenever a new pixel is scanned in, both the 1st and the 2nd accumulators update their values. The change in the 1st accumulator is to add in the new pixel value while the change in the 2nd accumulator is to add in the new output of the 1st accumulator, which is equal to the current sum of all scanned pixel value.

For k th pixel value which is scanned in a row with length N, it increases the output of the 1st accumulator by 0/1 when it is scanned in. However, since the 2nd accumulators will keep updating its value whenever the $N - k + 1$ leftover pixels are scanned in, the impact brought by the kth pixel will be repeated $N - k + 1$ times. If the pixel is 1, the output of 2nd accumulators will increase by $N - k + 1$.

Therefore, if the pixel at the end of the row is 1, it is added to the output of the 1st accumulator last. Such pixel increases the the output of the 1st by 1 and the output of the 2nd accumulators by 1. If there is no new pixel scanned in, data is not updated. While if the pixel at the beginning of the row is 1, the output of 2nd accumulators will increase by N.

Part b (3 points).

From part (a) we can see that if we input the value of rth row into the two accumulators while the pixel value is b_{1r}, b_{2r}, ..., b_{Nr} (from the first scanned pixel to the last), the output is equal to

$A2(r) = b_{N,r} + 2 * b_{N-1,r} + ... + ...(N - k + 1) * b_{k,r} + ... + N * b_{1,r}$. It is the first moment of rth row assuming the origin is at the position of the last scanned pixel.

If we use buffers to store the summation of each column(the results are from one accumulator), and feed the column addition into the device as in part (a), we could calculate the first moment of one image.

Or we can use buffers to store the first moment of each row (the output from the 2nd accumulator), clear the buffer and do the calculation for next row. Finally use one accumulator to calculate the summation of all buffered data.

Note, we can reuse the two accumulators here, thus we do not need another accumulators, which is not the case for part (c).

Part c (4 points).

We can add the third accumulator to the end of the second accumulator and it will add its current input to the current total when a new pixel is scanned in.

Consider again a row with length N.

For k th pixel value is scanned in, it increases the output of the 1st accumulator by 0/1.

If the pixel is 1 other than 0, the output changes of the 2nd accumulator will be 1, 2, ..., N-k+1 whenever the $N-k+1$ leftover pixels are scanned in one by one. The 3rd accumulators will keep recording the changing process of the 2nd accumulator output into its final output and the output will increase by $\sum_{i=1}^{i=N-k+1} i = (N-k+1)*(N-k+2)/2$.

If we input the value of rth row is b_{1r}, b_{2r}, ..., b_{Nr} (from the first pixel to the last), the output of the third accumulator is equal to $A3(r)|_{t=N} = b_{N,r} + 2*(2+1)/2*b_{N-1,r} + ... + (N-k+1)*(N-k+2)/2*b_{k,r} ... + N*(N+1)/2*b_{1,r}$.

We notice that

$$2*A3(r) - A2(r) = b_{N,r} + 2^2*b_{N-1,r} + ... + (N-k+1)^2*b_{k,r} ... + N^2*b_{1,r}$$

is the format of the second moments.

Therefore, once we use three connected accumulators and find out the $A3(r)$, $A2(r)$ value for each row and record them in the buffer, we can make use of one accumulators to calculate the summation of $2*A3(r) - A2(r)$ among all rows.

Or

We can notice that

$$\begin{aligned} A3(r)|_{t=N} &= b_{N,r} + 2*(2+1)/2*b_{N-1,r} + ... \\ &+ (N-k+1)*(N-k+2)/2*b_{k,r} + ... + N*(N+1)/2*b_{1,r} \\ A3(r)|_{t=N-1} &= b_{N-1,r} + 2*(2+1)/2*b_{N-2,r} + ... \\ &+ (N-k+1)*(N-k+2)/2*b_{k-1,r} + ... + (N-1)*N/2*b_{1,r} \end{aligned}$$

Thus,

$$A3(r)|_{t=N} + A3(r)|_{t=N-1} = b_{N,r} + 2^2 b_{N-1,r} + ... + k^2/b_{k,r} + ... + N^2*b_{1,r}$$

Then you can use storage to save the result of the third accumulator from the second to last result, and use one adder to add them up.

Note: There are many ways in doing this. The general principle is that we should not use multipliers since it will be very time consuming and we better use as much as possible parallel computation instead of serial computation.

PROBLEM 5b (10 points)

Let x_i be the x component of the centroid for ith image row, and σ_i be the standard deviation. It is known that

$$\sigma_i = k_i \varepsilon$$

Because the error is due only to effects at the ends of each row, the error in the centroid is independent of the length of a row. Thus, the error in the centroid for each image row is independent and the standard deviation should be the same, i.e., k_i is constant.

The final estimation of the x component of the centroid for whole image (d row) is to average all x_i as followed:

$$\bar{x} = \frac{1}{d} \sum_{i=1}^{d} x_i$$

Based on the statistical theory, we know that

$$\begin{aligned} var(\bar{x}) &= \frac{1}{d^2} \sum_{i=1}^{d} var(x_i) \\ &= \frac{1}{d^2} \sum_{i=1}^{d} \sigma_i^2 \\ &= \frac{1}{d^2} \sum_{i=1}^{d} (k\varepsilon)^2 \\ &= \frac{1}{d}(k\varepsilon)^2 \\ \sqrt{var(\bar{x})} &= \frac{1}{\sqrt{d}} k\varepsilon \end{aligned}$$

Thus, the standard deviation for estimated x component of the centroid is inversely proportional to the square root of image size.