Raleigh quotient $\dfrac{t^T S t}{t^T t}$ $\quad$ t_1, t_2, t_3 eigenvectors

$\lambda_1, \lambda_2, \lambda_3$ eigenvalues

$$r = \frac{\alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2 + \alpha_3^2 \lambda_3}{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} \quad \text{as small as possible}$$

$Z = -\dfrac{s \cdot t}{E_t}$ $\quad$ more reliable $\quad Z_{av} = \left(\iint \dfrac{s}{E_t} \right) \cdot t > 0$

sensitive to places where $E_t \approx 0$

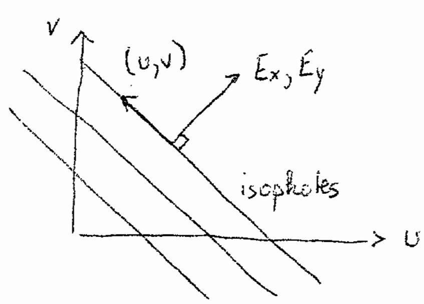
"stationary pixels"

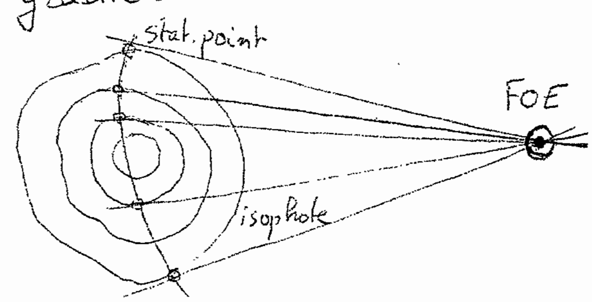
↳ non-changing brightness

$$u E_x + v E_y + E_t = 0$$

becomes $\quad u E_x + v E_y = 0 \quad$ or: $(u, v)^T \cdot (E_x, E_y) = 0$

→ motion field perpendicular to brightness gradient



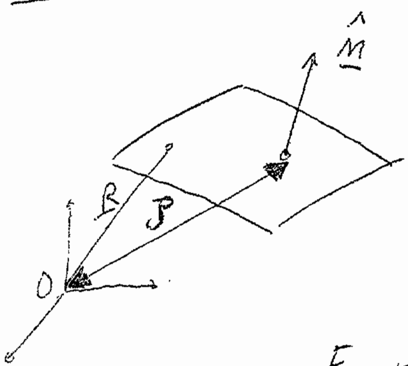


Take into account error in E_t:

$\dfrac{1}{E_t^2} \rightarrow \dfrac{1}{\varepsilon^2 + E_t^2}$ $\quad$ ε shall be not too small nor too large.

optimum: $\varepsilon^2 = \sigma^2$ of noise in E_t^2

__Example__ Planar Motion (allow $\underline{t}$ and $\underline{\omega}$, but in a plane)



Plane $= (\rho, \hat{\underline{n}})$

Equation: $\underline{R} \cdot \hat{\underline{n}} = \rho$ or $\underline{R} \cdot \underline{n} = 1$ where $\underline{n} = \dfrac{\hat{\underline{n}}}{\rho}$

Thus we take: Plane $= \left(\dfrac{1}{\rho}, \hat{\underline{n}}\right)$

$$E_t + \underline{v} \cdot \underline{\omega} + (\underline{r} \cdot \underline{n})(\underline{s} \cdot \underline{t}) = 0$$

unknowns: $\underline{t}, \underline{\omega}, \underline{n}$
→ 9 parameters
↳ 8 parameters

— scale factor ambiguity: $\underline{n} \to k\underline{n}$ $\underline{t} \to {}^1\!/\!_k \underline{t}$
↳ constraint: $\|\underline{t}\| = 1$ or $\|\underline{n}\| = 1$.

Next step: LSQ $\displaystyle \min_{\underline{\omega}, \underline{t}, \underline{n}} \iint \left(E_t + \underline{v}\,\underline{\omega} + (\underline{r} \cdot \underline{n})(\underline{s} \cdot \underline{t}) \right)^2$

$\partial \omega$: $\displaystyle \iint \left(E_t + v \cdot \underline{\omega} + (r \cdot n)(s \cdot t) \right) \underline{v} = 0$ → 3 equations → linear in ω, t, n

∂t: $\displaystyle \iint \left(E_t + \underline{v} \cdot \underline{\omega} + (r \cdot n)(s \cdot t) \right) \underline{(r \cdot n) s} = 0$ → 3 eq. → quadratic in n

∂n: $\displaystyle \iint \left(E_t + v \cdot \omega + (r \cdot n)(s \cdot t) \right) \underline{r (s \cdot t)} = 0$ → 3 eq. → quadratic in $\underline{t}$

Solving is not obvious.

Idea: iterative solution. Assume you know $\underline{t}$, and solve for $(\underline{\omega}, \underline{n})$
Then knowing $\underline{n}$, solve for $(\underline{\omega}, \underline{t})$.

It is proved to be stable (messy though)

__Uniqueness:__

$$E_t + \underline{v} \cdot \underline{\omega} + (\underline{r} \cdot \underline{n})(\underline{s} \cdot \underline{t}) = 0 \qquad (t, \omega, n)$$
$$E_t + \underline{v} \cdot \underline{\omega}' + (\underline{r} \cdot \underline{n}')(\underline{s} \cdot \underline{t}') = 0 \qquad (t', \omega', n')$$
$$\underline{v}(\underline{\omega} - \underline{\omega}') + (r \cdot n)(s \cdot t) - (r \cdot n')(s \cdot t') = 0$$

Idea: pull out $\underline{r}$ and $\underline{s}$ $(\underline{v} = \underline{r} \times \underline{s})$

Problem: $(\underline{r} \times \underline{s})(\underline{\omega} - \underline{\omega}') \ldots$

Isomorphism

vector $\longleftrightarrow$ skew symmetric matrix 3×3

$$\omega \times s = \Omega s \qquad \Omega = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix} \quad (\Omega^T = -\Omega)$$

Plugging-in : $\underline{r}^T \left[-(\Omega - \Omega') + \underline{m}\,\underline{t}^T - \underline{m}'\underline{t}'^T \right] \underline{s} = 0$

(at all point on the image, i.e for all $\underline{r}^T$)

Thus $\quad \left[-(\Omega - \Omega') + \underline{m}\, t^T - \underline{m}'\underline{t}'^T \right] \underline{s} = 0$

and also it's true for all textures $\underline{s}$.

Then $\quad -(\Omega - \Omega') + \underline{m}\,\underline{t}^T - \underline{m}'\underline{t}'^T = 0$

Transpose $\quad \Omega - \Omega' + t\, m^T - t' m'^T = 0$

$$\underline{m}\,\underline{t}^T + \underline{t}\,\underline{m}^T = \underline{m}'\underline{t}'^T + \underline{t}'^T \underline{m}^T$$

Trace $(\underline{m}.\underline{t}^T) = \underline{m}.\underline{t}$ $\quad$ thus : $\underline{m}.\underline{t} = \underline{m}'.\underline{t}'$

① $\|m'\| = 0$ or $\|t'\| = 0 \implies \|m\| = 0$ or $\|t\| = 0$

② $m' \,//\, m$ and $t' \,//\, t$ ~~and~~ $\|m'\|.\|t'\| = \|m\|.\|t\|$

③ $m' \,//\, t$ & $t' \,//\, m$ & $\|m'\|.\|t'\| = \|m\|.\|t\|$ (switch $\underline{m}$ and $\underline{t}$)

There aren't any other solutions! (Bezout told maybe more!)

Also : $\quad -(\Omega - \Omega') + (\underline{m}\,\underline{t}^T - \underline{t}\,\underline{m}^T) = 0$ implies

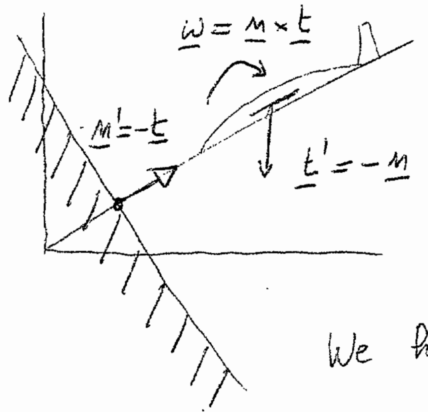
$$x \times (\omega - \omega') + x \times (\underline{m} \times \underline{t}) = 0 \quad \text{for any } x$$

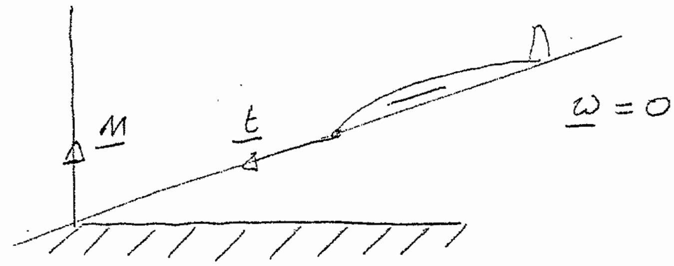
$$\underline{\omega}' - \underline{\omega} + \underline{m} \times \underline{t} = 0$$

$$\begin{cases} \underline{m}' = \underline{t} \\ \underline{t}' = \underline{m} \\ \underline{\omega}' = \underline{\omega} + \underline{m} \times \underline{t} \end{cases}$$

Does it happen in real life that we have two solutions?

Landing plane:





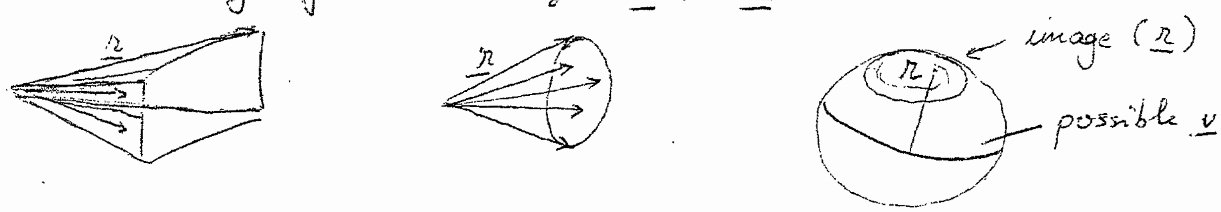
We have 2 different solutions here!

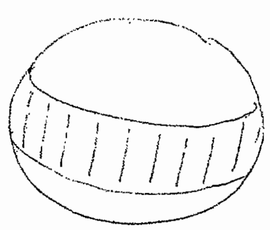
Stability

pure rotation $\iint \underline{v}\,\underline{v}^T$ condition number $\frac{\lambda_{max}}{\lambda_{min}}$

The larger CN is, the worse things are. $\det(\sum \underline{v}\,\underline{v}^T)$

What is the range of $\underline{v}$ vectors? $\underline{v} \perp \underline{r}$





permissible band

how wide is the band? answer: as wide as FOV.

$$\iint \underline{v}\,\underline{v}^t = \begin{pmatrix} 1+\frac{r_v^2}{2}+\frac{r_v^4}{6} & & 0 \\ & 1+\frac{r_v^2}{2}+\frac{r_v^4}{6} & \\ 0 & & r_v^2/2 \end{pmatrix} \qquad \frac{\lambda_{max}}{\lambda_{min}} \text{ BAD IF } r_v \simeq 0$$

Conclusion: bad for telephoto lense
good for wide-angle cameras