Engineering Risk Benefit Analysis

1.155, 2.943, 3.577, 6.938, 10.816, 13.621, 16.862, 22.82, ESD.72

CBA 2. The Time Value of Money

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The Importance of Time

- **A dollar in hand now is worth more than a dollar received in the future, because of its *earning power*, i.e., it can be invested to generate income.**

- **The *purchasing power* of money, i.e., the amount of goods that a certain amount can buy, changes with time also.**

- **Objective: To develop methods for establishing the *equivalence* of sums of money. It depends on the amounts, the time of occurrence of the sums of money, and the interest rate.**

Overview of Lecture

- The Basics of Interest Rates – Simple and Compound Interest
- The Basic Discount Factors – Present Value, Future Value, Annual Value
- Economic Equivalence and Net Present Value
- Return to Interest Rates: Nominal and Effective Rates
- Inflation

Simple Interest

- P: Principal amount
- n: Number of interest periods
- i: Interest rate
- I: Interest earned
- Interest <u>and</u> principal become due at the end of n.

$$I = Pni$$

- The interest is proportional to the length of time the principal amount was borrowed.

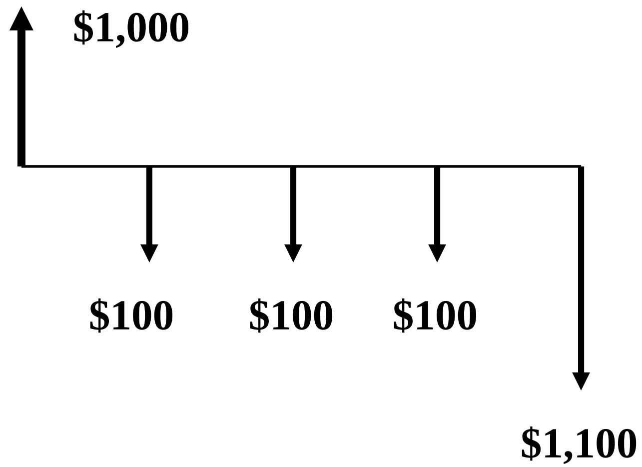
Compound Interest

- **Interest is payable at the end of each interest period.**
- **If the interest is not paid, the borrower is charged interest on the total amount owed (principal plus interest).**
- ***Example: $1,000 is borrowed for two years at 6% (compounded). A single payment will be made at the end of the second year.***
- **Amount owed at the beginning of year 2: $1,060**
- **Amount owed at the end of year 2:**

  $$\$1{,}060 \times 1.06 = \$1{,}000 \times (1.06)^2 = \$1{,}123.60$$

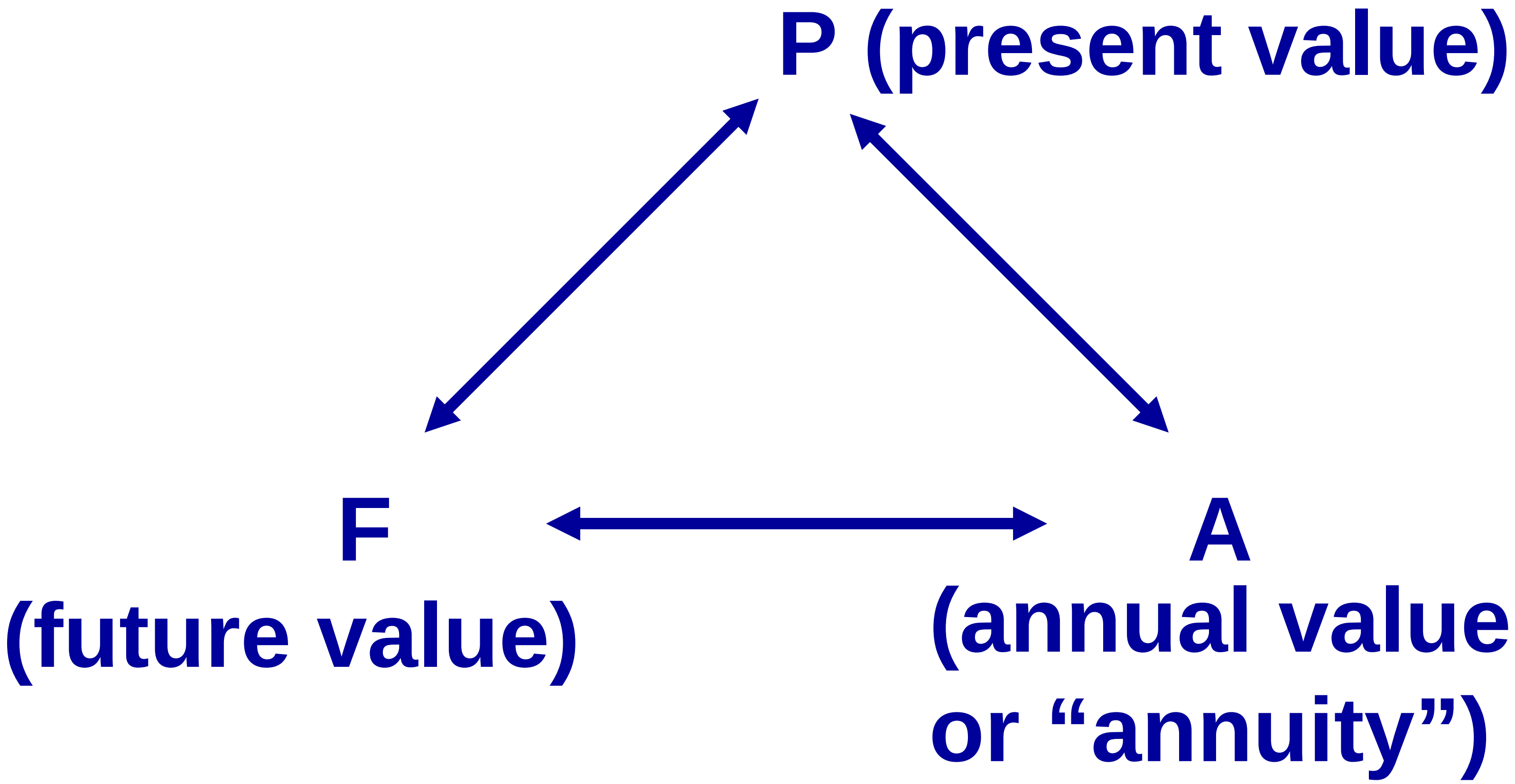
- **For simple interest, the amount owed at the end of year 2 would be: $1,000 + 1,000x2x0.06 = $1,120.00**

Cash Flows over Time

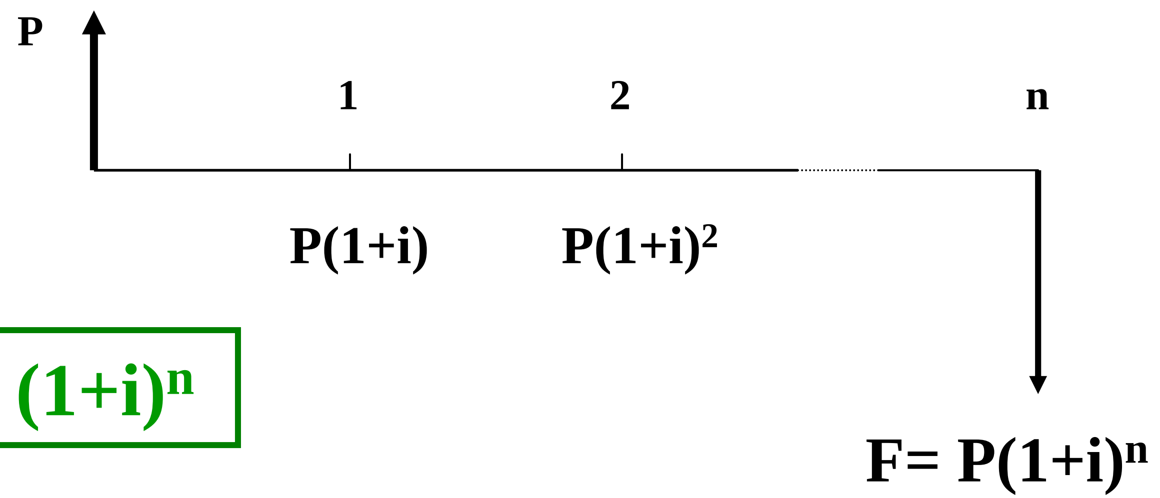


- Up arrow = we receive $; down arrow = we pay $
- Amount borrowed: $1,000
- Interest is paid at the end of each year at the rate of 10%.
- The principal is due at the end of the fourth year.

The Basic Discount Factors



Single-Payment Compound-Amount Factor



$(F/P, i, n) = (1+i)^n$

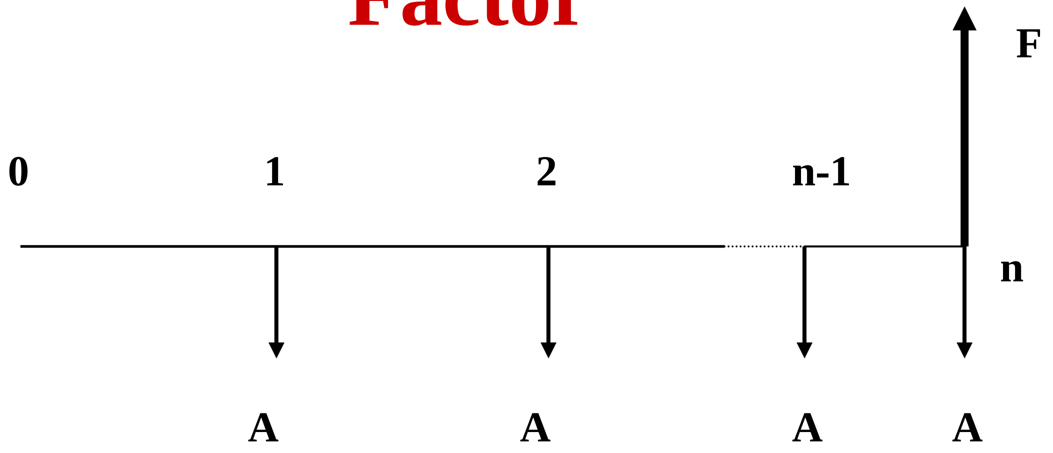
- **A single payment is made after n periods.**
- **The interest earned at the end of each period is charged on the total amount owed (principal plus interest).**
- **$1 now is worth (F/P,i,n) at time n if invested at i%**

Single-Payment Present-Worth Factor

$$(P / F, i, n) = \frac{1}{(1 + i)^n}$$

- The reciprocal of the single-payment compound amount factor.
- Discount rate: i
- $1 n years in the future is worth (P/F, i, n) now.

Equal-Payment-Series Compound-Amount Factor



- Equal payments, A, occur at the end of each period.
- We will get back (F/A, i, n) at the end of period n if funds are invested at an interest rate i.
- $F = A + A(1+i) + A(1+i)^2 + \ldots + A(1+i)^{n-1}$

$$(F/A,i,n) = \frac{(1+i)^n - 1}{i}$$

Equal-Payment-Series Sinking-Fund Factor

- **May be used to determine the payments A required to accumulate a future amount F.**

$$(A/F,i,n) = \frac{i}{(1+i)^n - 1}$$

Example. ***We wish to deposit an amount A every 6 months for 3 years so that we'll have $10,000 at the end of this period. The interest rate is 5% per year.***

n = 6 deposits i = 2.5% per 6-month period
F=$10,000 (A/F, 0.025, 6) = 0.15655 ⇒ A=$1,565.50

Equal-Payment-Series Capital-Recovery Factor (1)

- An amount P is deposited now at an annual interest rate i.
- We will withdraw the principal plus the interest in a series of equal annual amounts A over the next n years.
- The principal will be worth $P(1+i)^n$ (slide 8) at the end of n years. This amount is to be recovered by receiving A every year $\Rightarrow$ the sinking-factor formula applies (slide 11) $\Rightarrow$

Equal-Payment-Series Capital-Recovery Factor (2)

$$A = P(1+i)^n\left[\frac{i}{(1+i)^n - 1}\right] \quad \Rightarrow \quad \boxed{(A/P,i,n) = \frac{i(1+i)^n}{(1+i)^n - 1}}$$

Example: ***Your house mortgage is $300,000 for 30 years with an nominal annual rate of 7%. What is the monthly payment?***

n = 360 months **i = 0.583% per month**

(A/P, 0.00583, 360) = 0.006650339 $\Rightarrow$

A = 300,000x 0.006650339 = $1,995.10 per month

Summary of the Formulas

Single-Payment Compound-Amount Factor

$$(F/P, i, n) = (1+i)^n$$

Equal-Payment-Series Compound-Amount Factor

$$(F/A, i, n) = \frac{(1+i)^n - 1}{i}$$

Equal-Payment-Series Capital-Recovery Factor

$$(A/P, i, n) = \frac{i(1+i)^n}{(1+i)^n - 1}$$

Continuous Compounding (1)

- Suppose that interest is compounded a very large number of times. Then, the effective annual interest rate is

$$i = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m - 1 = e^r - 1$$

where r is the nominal annual interest rate.

Continuous Compounding (2)

$$(F/P, i, n) = (1+i)^n$$

$$(F/P, r, n) = e^{rn}$$

$$(F/A,i,n) = \frac{(1+i)^n - 1}{i}$$

$$(F/A,r,n) = \frac{e^{rn} - 1}{e^{r} - 1}$$

$$(A/P,i,n) = \frac{i(1+i)^n}{(1+i)^n - 1}$$

$$(A/P,r,n) = \frac{e^{r} - 1}{1 - e^{-rn}}$$

Nominal and Effective Interest Rates (1)

- ***The nominal interest rate*** **(or *annual percentage rate)* is the annual rate without the effect of any compounding.**
- ***The effective (actual) interest rate*** **is the annual rate taking into account the effect of any compounding during the year.**

<u>Example:</u> A credit card advertises a nominal rate of 18% compounded monthly. The actual rate is, then, (18/12) = 1.5% per month. The effective annual rate is

$(1.015)^{12} - 1 = 0.1956$ or 19.56%

(if you do not pay anything each month)

Nominal and Effective Interest Rates (2)

- **The effective interest rate i depends on the frequency of compounding.**
- **Example: nominal interest rate r = 10%**
  - ***Compounded annually: $i = r = 10\%$***
  - ***Compounded quarterly: $i = (1+0.1/4)^4 - 1 = 10.38\%$***
  - ***Compounded monthly: $i = (1+0.1/12)^{12} - 1 = 10.471\%$***
  - ***Compounded weekly: $i = (1+0.1/52)^{52} - 1 = 10.506\%$***
  - ***Compounded daily: $i = (1+0.1/365)^{365} - 1 = 10.516\%$***
  - ***Compounded continuously: $i = e^{0.1} - 1 = 10.517\%$***

Nominal and Effective Interest Rates (3)

In the formulas we introduced in earlier slides, i is the effective interest rate for a given period and n is the number of such periods.

<u>Example:</u> ***You wish to buy a house and you can afford to make a down payment of $50,000. Your monthly mortgage payment cannot exceed $2,000. If 30-year loans are available at 7.5% interest compounded monthly, what is the highest price that you may consider?***

Nominal and Effective Interest Rates (4)

Solution:

Let's use *one month* as the time period. Then, n = 360 months, and i = (7.5/12) = 0.625%. We know that

$$(A/P,i,n) = \frac{i(1+i)^n}{(1+i)^n - 1}$$

This yields (A/P, 0.00625, 360) = 0.00699 $\Rightarrow$

P x 0.00699 = (H - 50,000) x 0.00699 $\leq$ 2,000 $\Rightarrow$

H $\leq$ (2,000/0.00699) + 50,000 = $336,123

Nominal and Effective Interest Rates (5)

- **Let's use *one year* as the time period. Then, n = 30 years, and $i = (1+0.00625)^{12} - 1 = 7.763\%$**

Then, (A*/P, 0.07763, 30) = 0.0867 A* = 0.0867P per year

Your effective payment per year is

A* = $2,000x(F/A, 0.00625, 12) = $2,000x12.4212 = $24,842

P $\leq$ (24,842/0.0867) + 50,000 = $336,533 as before

Consistency between i and n will lead to identical solutions.

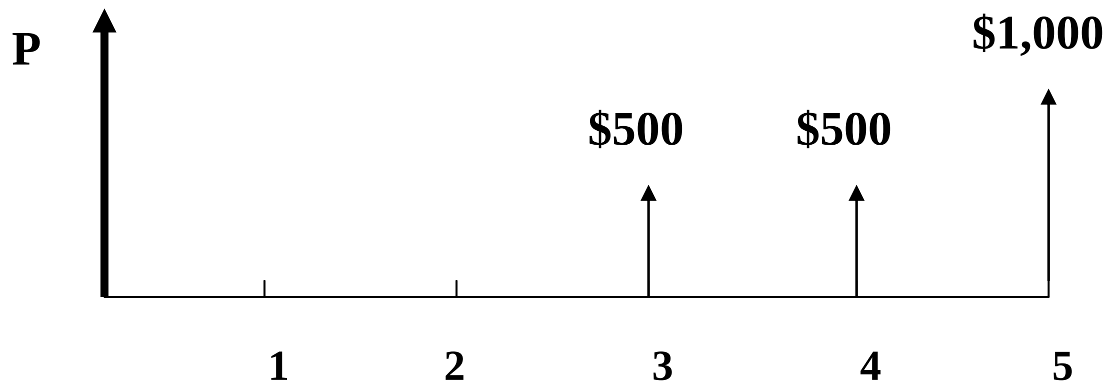
Economic Equivalence (1)

- **The formulas that we have developed establish economic equivalence between P and F, an equal-payment series and F, and so on.**

<u>Example:</u> ***Consider the following cash flow: You will receive $500 at the end of years 3 and 4 and $1,000 at the end of year 5. If the interest rate is 7%, what amount received at the present is equivalent to this cash flow?***

Economic Equivalence (2)

Solution:



$$P = \frac{500}{(1+0.07)^3} + \frac{500}{(1+0.07)^4} + \frac{1,000}{(1+0.07)^5}$$

P = 408.15 + 381.45 + 712.99 = $1,502.59

Economic Equivalence (3)

If the interest is compounded continuously, the result will be:

$$P = \frac{500}{e^{3x0.07}} + \frac{500}{e^{4x0.07}} + \frac{1{,}000}{e^{5x0.07}}$$

P = 405.30 + 377.89 + 704.69 = $1,487.88

Inflation

- **The *purchasing power* of money declines when the prices increase.**
- **This must be included in equivalence calculations.**
- **A *price index* is the ratio of the price of a commodity or service at some point in time to the price at some earlier point.**
- **The *Consumer Price Index (CPI)* represents the change in prices of a "market basket," that includes clothing, food, utilities, and transportation.**
- **The CPI measures the changes in retail prices to maintain a fixed standard of living for the "average" consumer.**

CPI and Inflation

Year	Consumer Price Index (CPI)	(Annual Rate of Inflation)
1967	100.0	2.9%
1972	125.3	3.3%
1977	181.5	6.5%
1980	246.8	13.5%
1985	322.2	3.6%
1990	391.4	5.4%
1995	456.5	2.8%
1999	497.6	1.9%

Figure by MIT OCW.

From Table 5.1 of Thuesen & Fabrycky, *Engineering Economy*, 7th Edition, Prentice Hall, NJ, 2001.

Inflation Rate

Annual inflation rate for year t+1:

$$f_{t+1} = \frac{CPI_{t+1} - CPI_t}{CPI_t}$$

For many calculations, an average inflation rate is sufficient.

$$CPI_t(1+f)^n = CPI_{t+n}$$

Note: Thuesen & Fabrycky use $\bar{f}$ for the average rate.

Example

The average inflation rate from 1967 to 1999 is given by

$$100(1+f)^{32} = 497.6 \quad \Rightarrow \quad 1+f = 4.976^{1/32} = 1.0514$$

$$\Rightarrow \quad f = 5.14\%$$

Definitions

- ***Market interest rate (or current-dollar interest rate)* i: The interest rate available in finance. Inflation impact is included.**
- ***Inflation-free interest rate (or constant-dollar interest rate)* i':**

  It represents the earning power of money with inflation removed. It must be calculated.
- ***Actual dollars*: The amount received or disbursed at any point in time.**
- ***Constant dollars*: The hypothetical amount received or disbursed in terms of the purchasing power of dollars at some base year.**

Constant and Actual Dollars

- **(actual dollars) = $(1+f)^n$ (constant dollars)**
  (based on the purchasing power n years earlier)
- **Equivalence in terms of actual dollars: Use i.**
- **Equivalence in terms of constant dollars: Use i'.**
- **Relationship among i, i', and f:**

$$i' = \frac{1+i}{1+f} - 1$$

Proof

<u>Proof:</u> At the base year (t=0), constant and actual dollars coincide.

Let P be the present value. Then, n years from now,

$F = (1+i)^n P$	actual dollars
$F' = (1+i')^n P$	constant dollars $\Rightarrow$
$F = (1+f)^n F' = (1+f)^n (1+i')^n P$	actual dollars $\Rightarrow$

$$i' = \frac{1+i}{1+f} - 1$$

Example: Going to the Movies

- ***1967 Ticket Price: $1.25***
- ***1999 Ticket Price: $8.50***
- ***Has there been a price increase above the rate of inflation?***

The average rate of inflation has been (slide 28):

f = 5.14%. The actual rate of increase is

$i = (8.5/1.25)^{1/32} - 1 = 0.0617$. Therefore,

$i' = [(1+0.0617)/(1+0.0514)] - 1 = 0.0098 \cong 1\%$

Example: Investments in Two Countries (1)

John has immigrated to the US where the inflation rate is 2% while his brother Joe has stayed in the old country where the inflation rate is 4.5%. The US banks give an interest rate of 5.5% while those of the old country give 8%.

1. What are the real interest rates in the two countries?

$$\mathbf{i}'_{\mathrm{US}} = \frac{1+0.055}{1+0.02} - 1 = 3.43\% \qquad \mathbf{i}'_{\mathrm{OC}} = \frac{1+0.08}{1+0.045} - 1 = 3.35\%$$

Example: Investments in Two Countries (2)

2. If John decides to invest in the Old Country, what would his real interest rate be?

Interest rate of the OC (John invests there)

$$i'_{US/OC} = \frac{1+0.08}{1+0.02} - 1 = 5.88\%$$

US inflation rate (John lives there)