Engineering Risk Benefit Analysis

1.155, 2.943, 3.577, 6.938, 10.816, 13.621, 16.862, 22.82, ESD.72

CBA 3. Bases for Comparison of Alternatives

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Overview

- **(Net) Present Worth or Value [(N)PW or (N)PV] and Annual Equivalent (AE)**
- **The Effect of Discount and Inflation Rates**
- **Using PW and AE as Decision Criteria**
- **The Internal Rate of Return as a Decision Criterion**
- **The Benefit-Cost Ratio as a Decision Criterion**
- **Comparing Projects with Unequal Lives**

Present Worth (1)

- ***Present Worth*: The net equivalent amount at the present that represents the difference between the equivalent receipts and the equivalent disbursements of an investment cash flow for a selected interest rate i.**

- **If F_t is the net cash flow at time t, then**

$$\mathbf{PW(i) = \sum_{t=0}^{n} F_t (1+i)^{-t}}$$

Present Worth (2)

- **Present worth may also refer to the present value of receipts (benefits) or disbursements (costs). In this case, the criterion for decision making is the *Net Present Worth.***
- **We will follow the definition on slide 3 (Thuesen & Fabrycky).**
- **Present Worth, Net Present Worth, and Net Present Value are equivalent terms.**

$$\mathbf{PW(i) = NPW(i) = NPV(i) = \sum_{t=0}^{t=n} B_t (1+i)^{-t} - \sum_{t=0}^{t=n} C_t (1+i)^{-t}}$$

Present Worth: Example

- *You buy a car and you put down \$5,000. Your payments will be \$500 per month for 3 years at a nominal interest rate of 10%. Assuming monthly compounding, what is the present price you are paying?*

From CBA 2, Slide 14, we get

$$\boxed{(P/A,i,n)=\frac{(1+i)^n-1}{i(1+i)^n}}$$

Here: A = \$500/mo, i = 10/12 = 0.83%, n = 36 months

$$P=5{,}000+500\left[\frac{(1+0.0083)^{36}-1}{0.0083(1+0.0083)^{36}}\right]=$$

$$=5{,}000+15{,}505=\$20{,}505$$

Annual Equivalent

- **The annual equivalent of receipts minus the annual equivalent of disbursements (the annualized profit).**
- **Any present worth can be converted to a series of equal annual amounts by multiplying by (A/P, i, n).**

$$\mathbf{AE(i) = PW(i)(A/P,i,n) = [\sum_{t=0}^{n} F_t(1+i)^{-t}][\frac{i(1+i)^n}{(1+i)^n - 1}]}$$

•Another name is *Net Annual Value (NAV)*.

•For fixed i and n, AE and PW yield the same results, i.e., the same ranking of alternatives.

PW and AE Criteria for Decision Making

- **An *investment alternative* A_j, j = 1,…,n, is a decision option representing a course of action.**
- **<u>Decision Criteria</u>:**

$$A_j \succ A_k \qquad \text{if} \qquad PW(i)_{Aj} > PW(i)_{Ak}$$

or

$$A_j \succ A_k \qquad \text{if} \qquad AE(i)_{Aj} > AE(i)_{Ak}$$

- **These criteria are using the *total* investment.**

Example of Total Investment Comparisons

End of Year	B_3	B_4
0	-$12,000	-$15,000
1	-1,200	-400
2	-1,200	-400
3	1,500	3,000

- The benefits from these alternatives are identical. We must select one.
- Assume that i = 10%.

Total Investment Comparisons (2)

$$PW(10\%)_{B3} = -12{,}000 - 1{,}200\frac{1.1^2 - 1}{0.1\text{x}1.1^2} + \frac{1{,}500}{1.1^3} =$$

$$= -12{,}000 - 1{,}200\text{x}1.736 + \frac{1{,}500}{1.331} = -\$12{,}956$$

$$PW(10\%)_{B4} = -\$13{,}440 < -\$12{,}956$$

- **Therefore, B_3 should be preferred over B_4.**

AE on Total Investment

$$AE(10\%)_{B3} = -12{,}000(A/P,10\%,3) - 1{,}200 + 2{,}700(A/F,10\%,3) =$$

$$= -12{,}000\frac{0.1x1.1^3}{1.1^3 - 1} - 1{,}200 + 2{,}700\frac{0.1}{1.1^3 - 1} =$$

$$= -12{,}000x0.402 - 1{,}200 + 2{,}700x0.302 = -\$5{,}209$$

$$AE(10\%)_{B4} = -15{,}000x0.402 - 400 + 3{,}400x0.302 =$$

$$= -\$5{,}403 < -\$5{,}209$$

Thus, B3 should be preferred over B4, just as before.

Impact of Inflation (1)

- **Suppose that the inflation rate is 9% and that the amounts shown on slide 8 are in terms of constant dollars.**

- **Converting them to actual dollars we get:**
- ➢ **End of year 1: F = 1,200(F/P, 9, 1) = $1{,}200 \times 1.09^1$ = $1,308**
- ➢ **End of year 2: $1{,}200 \times 1.09^2$ = 1,426**

End of Year	B_3	B_4
0	-$12,000	-$15,000
1	-1,308	-436
2	-1,426	-475
3	1,943	3,885

Impact of Inflation (2)

- **Recalculate the PWs.**

$$PW(10\%)_{B3} = -12{,}000 - \frac{1{,}308}{1.1} - \frac{1{,}426}{1.1^2} + \frac{1{,}943}{1.1^3} =$$

$$= -\$12{,}908 < -\$12{,}870 = PW(10\%)_{B4}$$

- **Therefore, B_4 is now preferred, while, in the case without inflation, B_3 was preferred (slide 9).**

PW and AE on Incremental Investment

- **Derive the difference between the two alternatives.**

Year	B_3	B_4	$B_4 - B_3$
0	-$12,000	-$15,000	-$3,000
1	-1,200	-400	800
2	-1,200	-400	800
3	1,500	3,000	1,500

- **If $PW(i)_{B4-B3} > 0$ or $AE(i)_{B4-B3} > 0 \Rightarrow B_4 \succ B_3$**

Example of Incremental Investment

$$PW(10\%)_{B_4-B_3} =$$

$$= -3{,}000 + 800(P/A,10,2) + 1{,}500(P/F,10,3) =$$

$$= -3{,}000 + 800\frac{1.1^2 - 1}{0.1x1.1^2} + \frac{1{,}500}{1.1^3} = -\$485 < 0$$

- B_3 should be accepted, just as in slide 9.
- PW- or AE-based results using total and incremental investments are identical.

Internal Rate of Return (IRR)

- **It is the interest rate for which the equivalent receipts of a cash flow equal the equivalent disbursements.**
- **It is the interest rate i* for which the present worth is zero.**

$$0 = PW(i^*) = \sum_{t=0}^{n} F_t (1 + i^*)^{-t}$$

- **The IRR represents the percentage or rate earned on the unrecovered balance of an investment such that the payment schedule makes the unrecovered investment equal to zero at the end of investment life.**

Example

End of year t	F_t
0	-1,000
1	-800
2	500
3	500
4	500
5	1,200

$$0 = PW(i^*) =$$

$$= -1000 - 800(P/F, i^*, 1) + 500(P/A, i^*, 4)(P/F, i^*, 1) +$$

$$+ 700(P/F, i^*, 5)$$

Example (cont'd)

Trial and error:

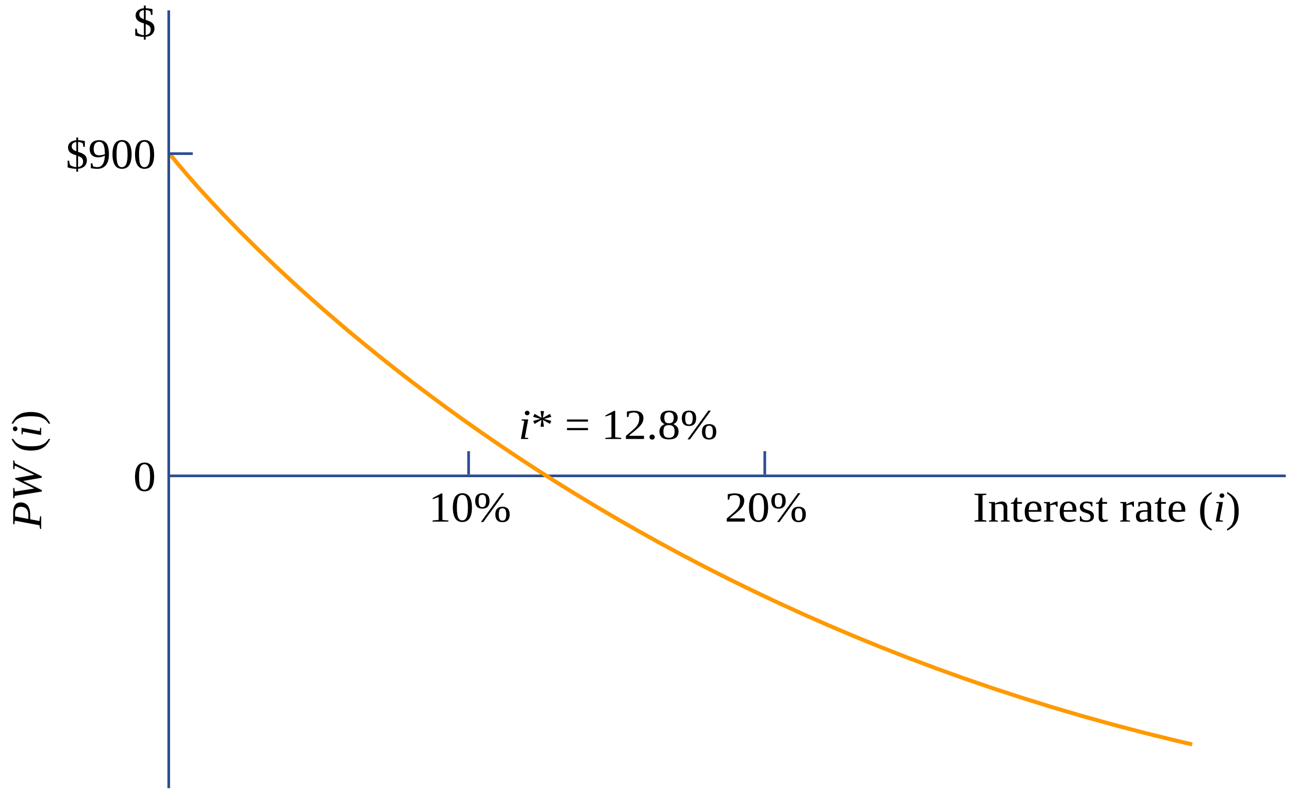
For i* = 12% ⇒ PW(12) = $39

For i* = 13% ⇒ PW(13) = -$12

By interpolation:

$$i^* = 12\% + 1\% \frac{39 - 0}{39 - (-12)} = 12.8\%$$

IRR and PW(i)



IRR and its relationship to the present-worth amount.

Figure by MIT OCW.

Thuesen & Fabrycky, 9th Edition

Cash Flows with a Single IRR

- The IRR is a useful concept when the shape of PW(i) is like the one on Fig. 6.3 (slide 18).

<u>Sufficient Conditions</u>

1. $F_0 < 0$ (The first nonzero cash flow is a disbursement)
2. The sequence $F_0, F_1, F_2, \ldots, F_n$, has one change in sign only.
3. $PW(0) > 0$ (sum of all receipts > sum of all disbursements)

Examples

- Both B_3 and B_4 of slide 8 fail the third condition. For B_3, PW(0) = -$12,900 <0.
- For the cash flow on slide 16, PW(0) = 900 > 0.

- Consider the cash flow:

t	0	1	2	3	4	5
$	-1,000	+600	+600	+600	+600	-1,410

  - The 2nd and 3rd conditions are violated.
  - Two IRRs are obtained: 1% and 18.5%
  - The IRR should not be used in decision making.

Minimum Attractive Rate of Return (MARR)

- **MARR is a cut-off rate representing a yield on investments that is considered minimally acceptable.**
- **The investor can *always* receive this rate (e.g., it could be the bank rate).**
- **It is determined by senior management.**
  - **If MARR is too high: Opportunities are lost.**
  - **If MARR is too low: Income is lost.**

The Do-Nothing Alternative

- **The investor will do nothing about the proposed alternatives and the funds will be placed in investments that yield an IRR equal to the MARR.**

- **$PW(MARR)_{A0} = 0$**
  $AE(MARR)_{A0} = 0$

- **For computational purposes, we may assume that the cash flows for A_0 alternative are all zero.**

IRR on Incremental Investment

1. Make sure the cash flows satisfy the conditions on slide 19.
2. List alternatives in ascending order based on initial cost.
3. The “current best” alternative can be the “Do Nothing” one.
4. Determine the differences between the “challenging” alternative and the current best alternative.
5. If $i^*_{A_k - A_j} > \text{MARR} \quad \Rightarrow \quad A_k \succ A_j$

Example (1)

End of Year	A0	A1	A2	A3
0	0	-$5,000	-8,000	-10,000
1 – 10	0	1,400	1,900	2,500

MARR = 15%

1. All satisfy the conditions on slide 19.
2. Find $i^*_{A1-A0} \equiv x_{10}$

Example (2)

$$0 = -5{,}000 + 1{,}400(P/A, x_{10}, 10)$$

$$0 = -5{,}000 + 1{,}400\frac{(1+x_{10})^{10}-1}{x_{10}(1+x_{10})^{10}}$$

$$i^*_{A_1-A_0} \equiv x_{10} = 25\% > 15\%$$

Therefore, A_1 replaces A_0 as the current best alternative.

Example (3)

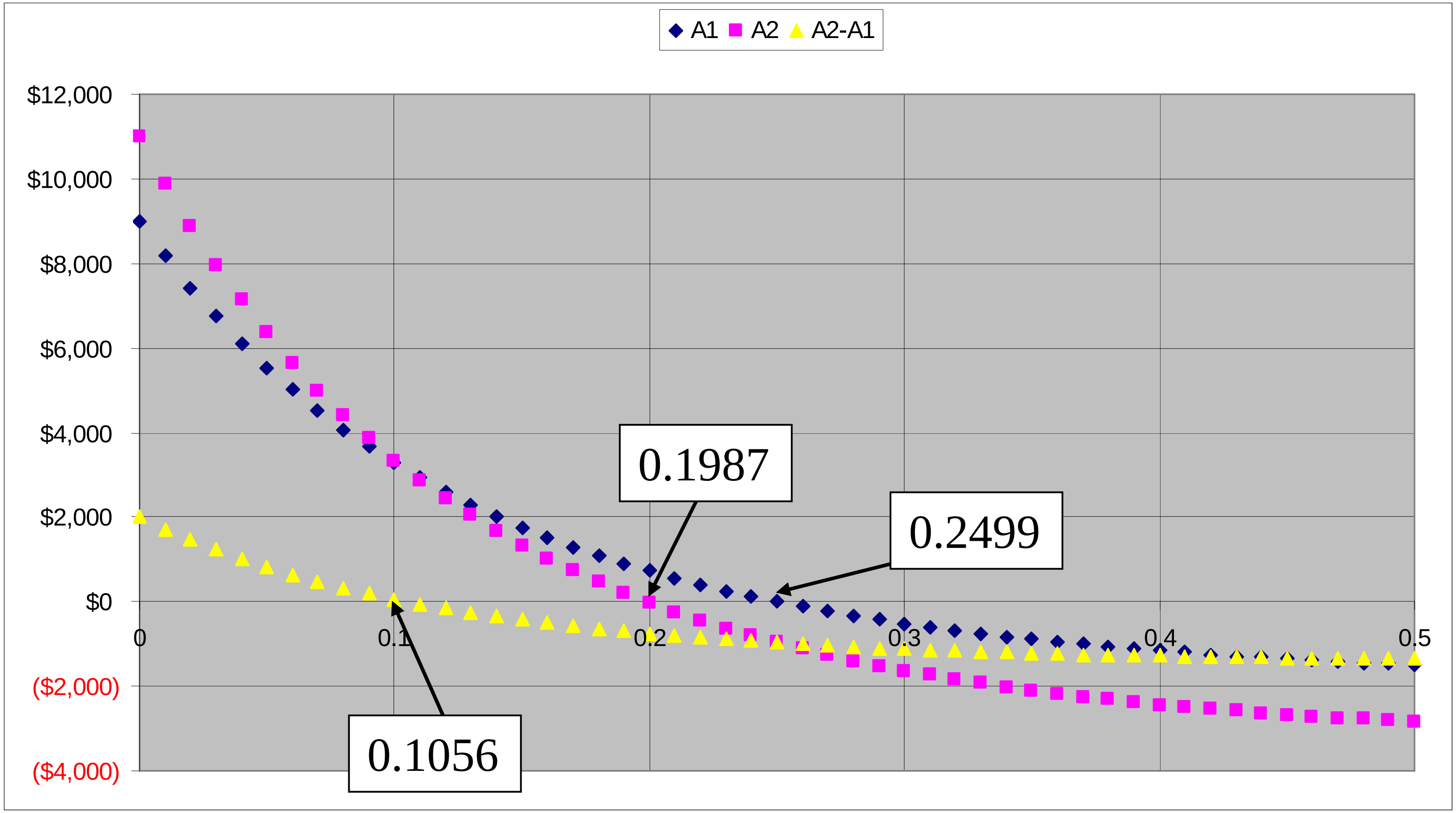
Similarly, we solve $0 = -3{,}000 + 500(P/A, x_{21}, 10)$

$$0 = -3{,}000 + 500\frac{(1+x_{21})^{10} - 1}{x_{21}(1+x_{21})^{10}}$$

$$i^*_{A_2 - A_1} \equiv x_{21} = 10.5\% < 15\%$$

Therefore, A_1 remains the current best alternative.

Example (4)



A1
A2
A2-A1
$12,000
$10,000
$8,000
$6,000
$4,000
$2,000
$0
($2,000)
($4,000)
0
0.1
0.2
0.3
0.4
0.5
0.1987
0.2499
0.1056

Example (5)

To compare A_3 to A_1, we must solve

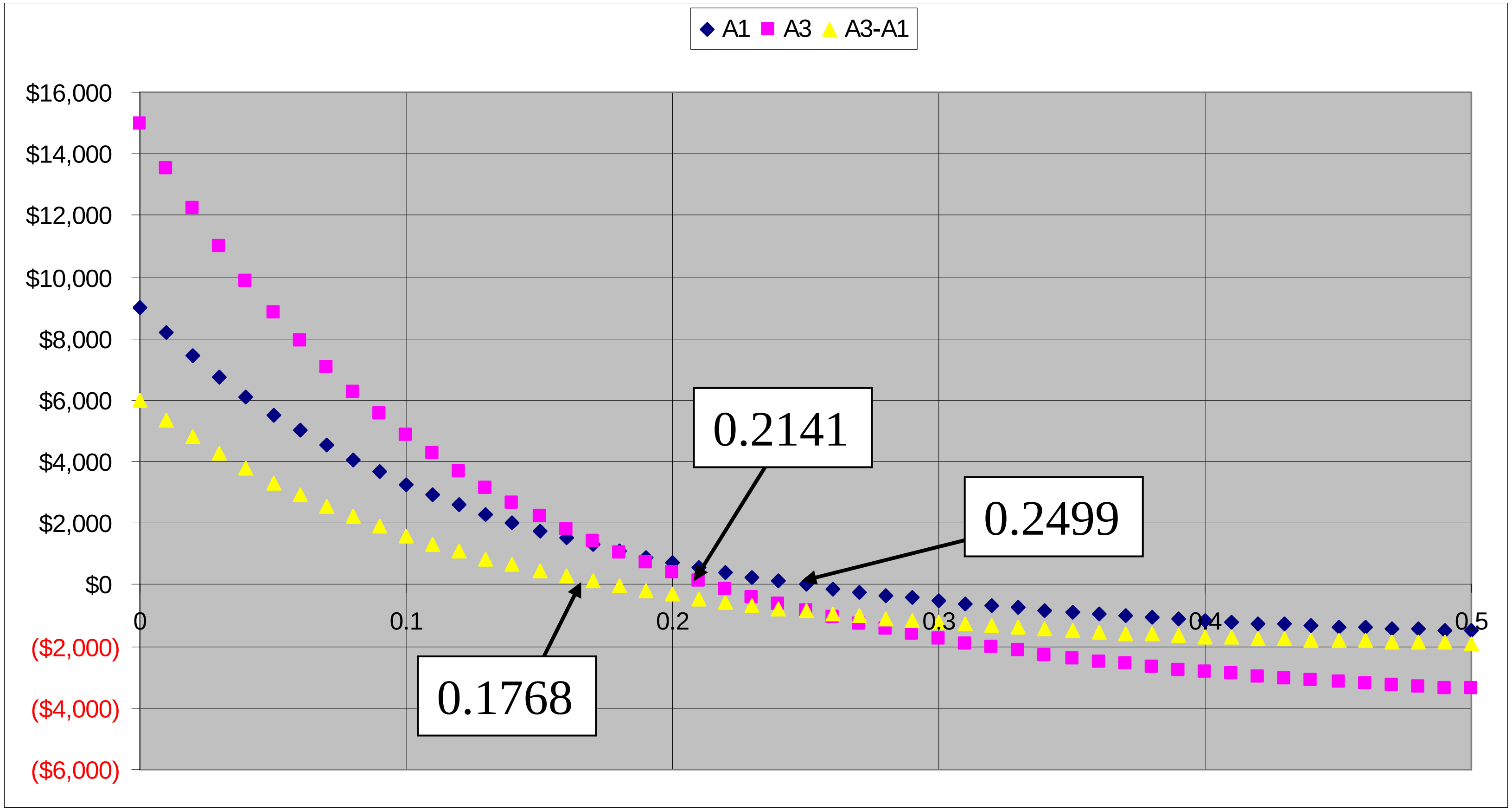
0 = -5,000 + 1,100(P/A, x_{31}, 10)

$$0 = -5{,}000 + 1{,}100\frac{(1+x_{31})^{10}-1}{x_{31}(1+x_{31})^{10}}$$

$$i^*_{A3-A1} \equiv x_{31} = 17.6\% > 15\%$$

Therefore, A_3 becomes the best solution.

Example (6)



A1
A3
A3-A1
$16,000
$14,000
$12,000
$10,000
$8,000
$6,000
$4,000
$2,000
$0
($2,000)
($4,000)
($6,000)
0
0.1
0.2
0.3
0.4
0.5
0.2141
0.2499
0.1768

IRR on Total Investment (1)

- We can calculate the IRRs for the total cash flows on slide 24.

- Let $x_2 \equiv i^*_{A_2}$ then

$$0 = -8{,}000 + 1{,}900 \frac{(1+x_2)^{10} - 1}{x_2(1+x_2)^{10}}$$

$$x_2 \equiv i^*_{A_2} = 19.9\%$$

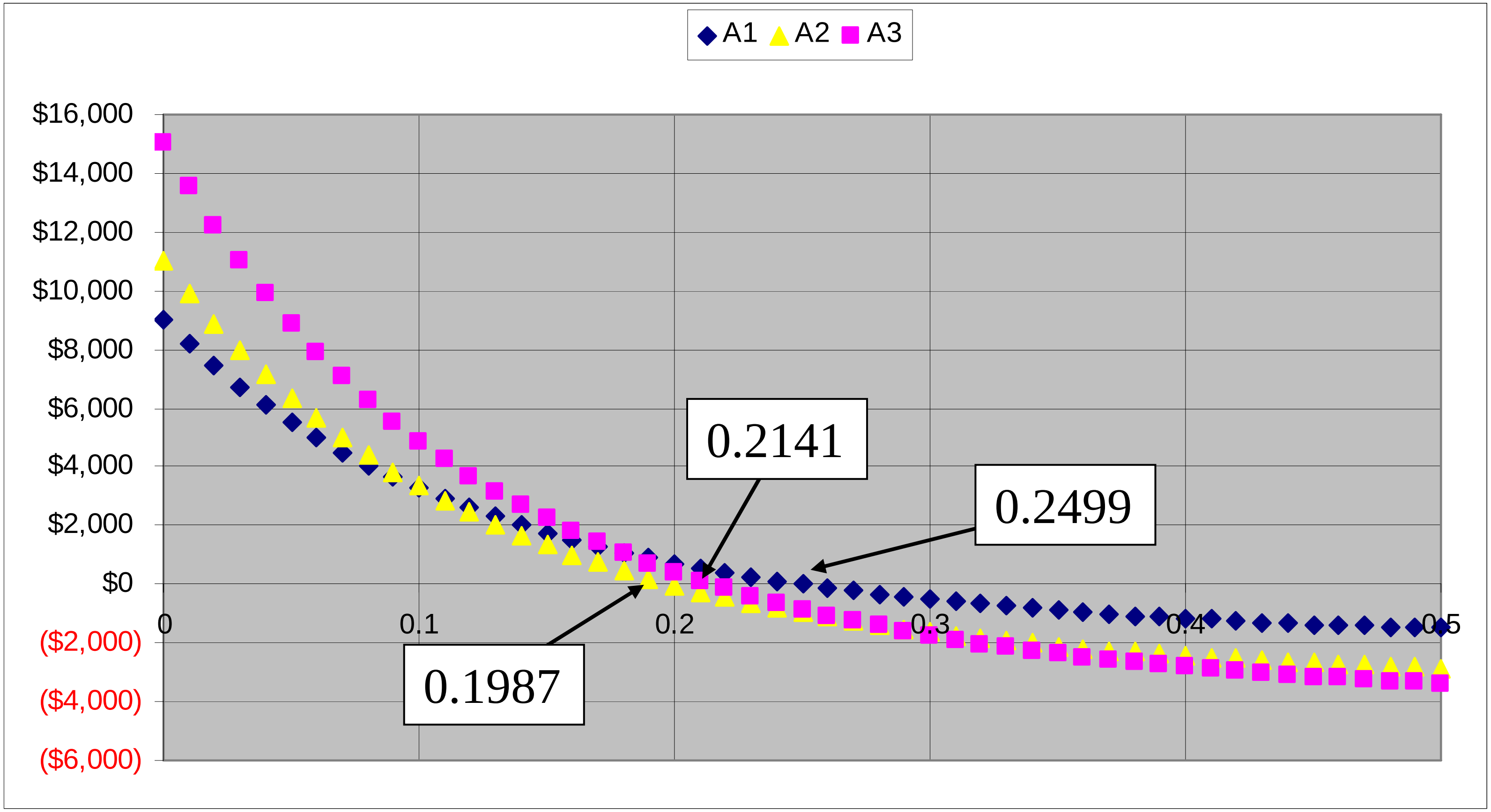
IRR on Total Investment (2)

Similarly, $\mathbf{i^*_{A0} = 15\%}$ $\mathbf{i^*_{A1} = 25\%}$

$\mathbf{i^*_{A2} = 19.9\%}$ $\mathbf{i^*_{A_3} = 21.4\%}$

- **The "best" alternative (i.e., the one having the highest IRR) is A_1, not A_3 (slide 28).**

IRR on Total Investment (3)



PW on Incremental Investment Revisited

We have already investigated this method on slide 14. The major result was that:

- **PW- (or AE-)based results using total and incremental investments are identical.**

Approach

1. **Make sure the cash flows satisfy the conditions on slide 19.**
2. **List alternatives in ascending order based on initial cost.**

PW on Incremental Investment Revisited (2)

3. **The "current best" alternative can be the Do Nothing one.**
4. **Determine the differences between the "challenging" alternative (next highest initial cost) and the current best alternative.**
5. **If** $\mathbf{PW(MARR)_{A_k - A_j} > 0 \quad \Rightarrow \quad A_k \succ A_j}$

The Example Revisited (1)

- **Consider again the cash flows on slide 24.**
- **$PW(15)_{A1\text{-}A0}$ = -5,000 + 1,400(P/A, 15, 10)** $\Rightarrow$

$$PW(15)_{A_1-A_0} = -5{,}000 + 1{,}400\frac{(1+0.15)^{10}-1}{0.15(1+0.15)^{10}} =$$

$$= 2{,}026 > 0$$

- **Therefore, A_1 becomes the current best alternative.**

The Example Revisited (2)

Similarly, $PW(15)_{A2\text{-}A1}$ = -3,000 + 500(P/A, 15, 10)

$$PW(15)_{A_2-A_1} = -3{,}000 + 500\,\frac{(1+0.15)^{10}-1}{0.15(1+0.15)^{10}} =$$

$$= -490 < 0$$

- Therefore, A_1 remains the current best alternative.

The Example Revisited (3)

- **Finally,**

$$\mathbf{PW(15)_{A_3-A_1} = -5{,}000 + 1{,}100\frac{(1+0.15)^{10}-1}{0.15(1+0.15)^{10}} =}$$

$$\mathbf{= 521 > 0}$$

- **Therefore, A_3 becomes the current <u>and final</u> best alternative. This is the same as in slide 28, but not the result on slide 31 (best: A_1).**

Summary

- **The economic desirability of alternatives is determined by examining their *differences*.**
- **Use the PW (or, equivalently, the AE) criterion, as described on slides 32 – 33.**
- **PW- (or AE-)based results using total and incremental investments are identical.**
- **The IRR criterion (assuming that the conditions on slide 18 are satisfied) on the differences gives results consistent with those of the PW and AE criteria.**
- **The IRR criterion on total investments may not give results consistent with those of the PW and AE criteria.**

Unequal Lives

- **All alternatives must be compared over the same time span.**
- **Assumptions are required to compare them over the same *study period* or *planning horizon*.**

Example (1)

Alternative	A	B	C
Initial Cost	$4,000	$16,000	$20,000
Annual Cost	$6,400	$1,400	$1,000
Lifetime	6 years	3 years	4 years

A, B, and C all fulfill the same objective, but for a different number of years; select the least costly for i = 7%

Example (2)

- AE(A) = 6.4K + 4K (A/P, 7%, 6) ≈ 6.4K + 4K (0.21) = $7,240

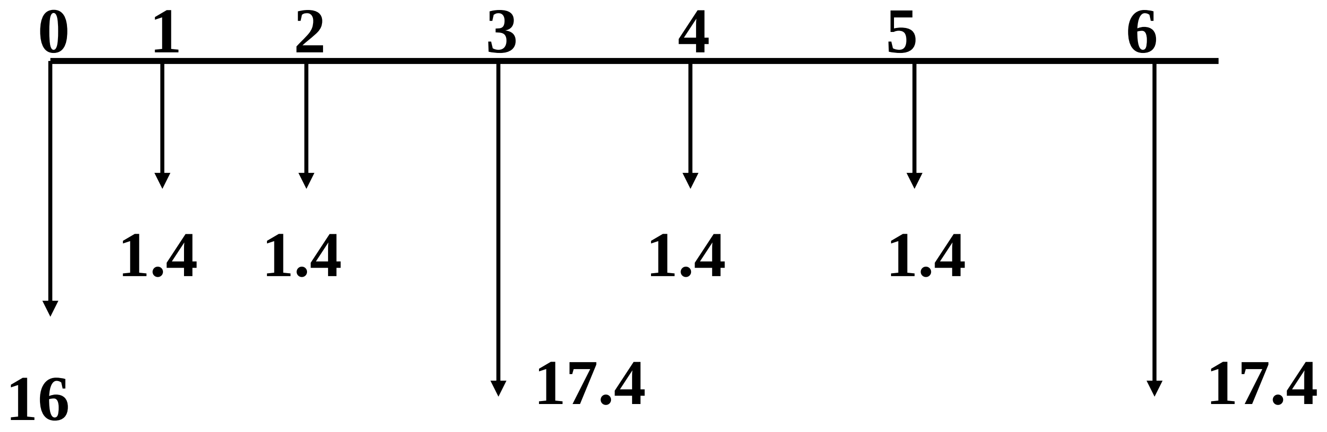
- AE(B) = 1.4K + 16K(A/P, 7%, 3) ≈ 1.4K+16K (0.38) = $7,480

- AE(C) = 1K + 20K (A/P, 7%, 4) ≈ 1K+ 20K (0.3) = = $7,000

Repeating Cash Flows (1)

- **When sequences of cash flows are repeated, it is only necessary to calculate the AE for the first sequence to find the AE for all sequences.**

- **For Alternative B, consider the extended B*:**



Repeating Cash Flows (2)

- **AE(B*) = 1.4 + 16 [1 + (P/F, 0.07, 3)] (A/P, 0.07, 6) = 1.4 + 16 x 1.816 x 0.21 = 1.4 + 6.10 = 7.50K**

  which is (approximately) the same as the original value of 7.48K.

- **The AEs that we calculated in slide 41 represent the following extended cash flows.**

Extended Cash Flows

t (years)	A (in $000)	B (in $000)	C (in $000)
0	4	16	20
1	6.4	1.4	1
2	6.4	1.4	1
3	6.4	17.4	1
4	6.4	1.4	21
5	6.4	1.4	1
6	10.4	17.4	1
7	6.4	1.4	1
8	6.4	1.4	21
9	6.4	17.4	1
10	6.4	1.4	1
11	6.4	1.4	1
12	6.4	1.4	1

Present Worth

- **PW(A, 12 yrs) = 4K + 6.4K (P/A, 7%, 12)**
+ 4K (P/F, 7%, 6) ≈ $57,500

- **PW(B, 12 yrs) = 16K + 1.4K (P/A, 7%, 12) + 16K (P/F, 7%, 3) + 16K (P/F, 7%, 6) + 16K (P/F, 7%, 9)≈$59,400**

- **PW(C, 12 yrs) = 20K + 1K (P/A, 7%, 12)**
+ 20K (P/F, 7%, 4) + 20K (P/F, 7%, 8) ≈ $55,600

- **These are PWs of costs over 12 years – the lowest common multiple of the lifetimes.**

- **Best alternative: C**

Present Worth of Original Alternatives

- **PW(A, 6 yrs) = 4 + 6.4(P/A, 7%, 6) = 4 +(6.4)(4.76) = = $34.46K**
- **PW(B, 3 yrs) = 16 + 1.4 (P/A, 7%, 3) = = 16 +(1.4)(2.62) = $19.68K**
- **PW(C, 4 yrs) = 20 + 1 (P/A, 7%, 4) = 20 + 1(3.4) = = $23.4K**

- **These are the PWs of costs over the actual lifetimes.**

- **Best alternative: B (not C). Incorrect.**