

Derivations in the Sentential Calculus

Having already two methods for showing an SC argument to be valid, the method of truth tables and search-for-counterexamples, we now want to introduce a third, in which we show an argument to be valid by producing a *proof*, which is a sequence of sentences formed according to certain rules that guarantee that any sentence that the rules permit you to write down is a logical consequence of a specified set of premisses. It may seem a little perverse to introduce a new technique when we have two already, but the actual situation is even worse than this, for the new procedure is, in a crucial respect, vastly inferior to the ones we have already. The old methods were *decision procedures*, which enabled us to test whether a given argument was valid. If the argument was valid, the procedure would show its validity, whereas if the argument was invalid, the procedure would show its invalidity. The new method is only a *proof procedure*: If an argument is valid, the method will enable to show that it is valid, but the method won't provide us any way of showing that an invalid argument is invalid. If an argument is valid, we can show it's valid by providing a proof, but the fact that we haven't been able to find a proof for a given argument doesn't show that the argument is invalid; maybe we just haven't looked hard enough.

The new method is a giant step backward. The reason for taking it will only appear when we start work on the predicate calculus. For predicate calculus, there is no decision procedure — the proof of this is one of the crowning glories of Logic II — but there is a proof procedure, and the proof procedure we will eventually learn for the predicate calculus will closely resemble the proof procedure we will now learn for the sentential calculus. Learning the proof procedure now will make our task later easier.

There are a great number of systems of proof in use. The particular system we shall study here was developed by Benson Mates in his superb textbook, *Elementary Logic* (2nd ed. New York: Oxford University Press, 1972).

A *proof* or *derivation* consists of a consecutively numbered sequence of sentences. The number of a sentence, which is written directly to its left, is its *line number*. To the left of the line number for the n th line is a (possibly empty) sequence of numbers $\leq n$. These are the *premiss numbers* of the n th line. The sentences whose line numbers are the premiss numbers of the n th line constitute the *premiss set* of the n th line. Sentences are written down in accordance with rules that make sure that each sentence we write down at any stage of a derivation is a logical consequence of its premiss set. If we introduce a new rule, what we have to make sure of is this: If we use the new rule to add a new line to a derivation that up till now has had the property that each line was a logical consequence of its premiss set, the new line will likewise be a logical consequence of its premiss set. If all our rules have this property, we can be confident that any sentence we write down at any stage of a derivation will be a logical consequence of its premiss set.

Before giving the rules, let me write down a derivation, so that you can see what an SC derivation looks like. The rules will come later.

1	1. $\neg\neg P$	PI
1	2. $(\neg\neg\neg\neg P \rightarrow \neg\neg P)$	CP, 1
3	3. $\neg P$	PI
2,3	4. $\neg\neg\neg P$	MT, 2, 3
2	5. $(\neg P \rightarrow \neg\neg\neg P)$	CP, 3, 4
2	6. P	MT, 1, 5
TH1	7. $(\neg\neg P \rightarrow P)$	CP, 1, 6

The boldface “**TH1**” in the bottom left isn’t a premiss number, and, in fact, it isn’t a part of the derivation at all. It’s a bookkeeping notation for keeping track of the things we’ve proved. I am keeping a tally, which is on the last page of this chapter.

Now for the rules. The governing principle in devising the rules is to make sure that, at each stage of a derivation according to the rules, the sentence we write down is a logical consequence of its premiss set. The first four rules only involve the connectives “ \rightarrow ” and “ \neg ,” and they suffice to capture all the valid arguments that only involve these two connectives. If Γ is a set of SC sentences and ϕ an SC sentence that contain no connectives other than “ \rightarrow ” and “ \neg ,” ϕ is a logical consequence of Γ if and only if there is a derivation by the four rules of ϕ whose premiss set is included in Γ . Thus the derivation above will show that “ $(\neg\neg P \rightarrow P)$ ” is a logical consequence of the empty set, that it, that it is valid. Later, we’ll add rules for the other connectives.

Premiss Introduction Rule (PI). You may write down any sentence you like if you take the sentence as its own premiss set.

Obviously, anything you write down by rule PI will be a logical consequence of its premiss set, since a sentence is a consequence of itself. One use of rule PI is simply to let us write down the premisses of our argument. Other, more technical uses, will emerge.

Our next rule gives us a method for proving a conditional: To prove a conditional, assume the antecedent as a premiss, then try to derive the consequent. This will be the central strategy for almost all our proofs.

Conditional Proof Rule (CP). If you have derived ψ with premiss set Γ , you may write $(\phi \rightarrow \psi)$ with premiss set $\Gamma \sim \{\phi\}$.

Rule CP is logical-consequence preserving, since, if ψ is a logical consequence of Γ , then $(\phi \rightarrow \psi)$ is a logical consequence of $\Gamma \sim \{\phi\}$. Most of the time, when we apply CP, the antecedent ϕ will be an element of the premiss set Γ , and the effect of the rule is to reduce the size of the premiss set. Occasionally, we’ll apply the rule when ϕ isn’t an element of Γ ; this is what we did in line 2 of the derivation above. Such applications are legitimated by the fact that a conditional is a consequence of its consequent.

A simpler illustration of the two ways of employing CP is the following derivation of “ $(P \rightarrow (Q \rightarrow P))$ ”:

1	1. Q	PI
1	2. $(P \rightarrow Q)$	CP, 1
TH2	3. $(Q \rightarrow (P \rightarrow Q))$	CP, 1, 2

Sentences like this that we derive from the empty set are called *SC theorems* or *theorems of logic*. Any such theorem is logically valid.

Modus Ponens Rule (MP). If you have derived ϕ with premiss set Γ and $(\phi \rightarrow \psi)$ with premiss set Δ , you may write ψ with premiss set $\Gamma \cup \Delta$.

Modus Tollens Rule (MT). If you have derived ψ with premiss set Γ and $(\neg \phi \rightarrow \neg \psi)$ with premiss set Δ , you may write ϕ with premiss set $\Gamma \cup \Delta$.

Clearly, if ϕ is a logical consequence of Γ and $(\phi \rightarrow \psi)$ is a consequence of Δ , ψ is a consequence of $\Gamma \cup \Delta$, and furthermore, if ψ is a consequence of Γ and $(\neg \phi \rightarrow \neg \psi)$ is a consequence of Δ , ϕ will be a consequence of $\Gamma \cup \Delta$.

As an example, let us derive the so-called principle of the syllogism, “ $((P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R)))$,” from the empty premiss set.

1	1. $(P \rightarrow Q)$	PI
2	2. $(Q \rightarrow R)$	PI
3	3. P	PI
1,3	4. Q	MP, 1, 3
1,2,3	5. R	MP, 2, 4
1,2	6. $(P \rightarrow R)$	CP, 3, 5
1	7. $((Q \rightarrow R) \rightarrow (P \rightarrow R))$	CP, 2, 6
TH3	8. $((P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R)))$	CP, 1, 7

This derivation illustrates our standard strategy for proving conditionals: Assume the antecedent and try to derive the consequent. Once you have it, you can use CP to remove the antecedent from the premiss set. As another example of the same strategy, let’s derive “ $((P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R)))$ ” from the empty premiss set. We begin by assuming the antecedent, hoping to fill the missing space in the following derivation:

1	1. $(P \rightarrow (Q \rightarrow R))$	PI
.....		
1	n. $(Q \rightarrow (P \rightarrow R))$	
	n+1. $((P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R)))$	CP, 1, n

Again, the thing we're trying to prove is a conditional, and we prove it by assuming the antecedent and deriving the consequent:

1	1.	$(P \rightarrow (Q \rightarrow R))$	PI
2	2.	Q	PI
.....			
1,2	n-1.	$(P \rightarrow R)$	
1	n.	$(Q \rightarrow (P \rightarrow R))$	CP, 2, n-1
	n+1.	$((P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R)))$	CP, 1, n

In trying prove " $(P \rightarrow R)$," we follow the standard strategy, assuming " P " and trying to derive " R ":

1	1.	$(P \rightarrow (Q \rightarrow R))$	PI
2	2.	Q	PI
3	3.	P	PI
.....			
1,2,3	n-2.	R	
1,2	n-1.	$(P \rightarrow R)$	CP, 3, n-2
1	n.	$(Q \rightarrow (P \rightarrow R))$	CP, 2, n-1
	n+1	$((P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R)))$	CP, 1, n

Now that we've gotten this far, we can complete the proof by filling in a couple of steps of *modus ponens*:

1	1.	$(P \rightarrow (Q \rightarrow R))$	PI
2	2.	Q	PI
3	3.	P	PI
1,2	4.	$(Q \rightarrow R)$	MP, 1, 3
1,2,3	5.	R	MP, 2, 4
1,2	6.	$(P \rightarrow R)$	CP, 3, 5
1	7.	$(Q \rightarrow (P \rightarrow R))$	CP, 2, 6
TH4	8.	$((P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R)))$	CP, 1

Our first derivation was the principle of double negation elimination, " $(\neg \neg P \rightarrow P)$." Now let's derive the converse principle of double negation introduction:

1	1.	P	PI
2	2.	$\neg \neg \neg P$	PI
2	3.	$(\neg \neg \neg \neg P \rightarrow \neg \neg \neg P)$	CP, 2
4	4.	$\neg \neg P$	PI
2,4	5.	$\neg \neg \neg \neg P$	MT, 3, 4
2	6.	$(\neg \neg P \rightarrow \neg \neg \neg \neg P)$	CP, 4, 5
2	7.	$\neg P$	MT, 2, 6
	h.	$(\neg \neg \neg P \rightarrow \neg P)$	CP, 2, 7
1	9.	$\neg \neg P$	MT, 1, 8
TH5	10.	$(P \rightarrow \neg \neg P)$	

Looking at this proof, we see that lines 2 through 9 are exactly like the proof of TH1, except that “P” has been replaced everywhere by “¬P.” This exemplifies a general phenomenon. Whenever we have a substitution s and a derivation of φ from the empty set, we can get a derivation of $s(\varphi)$ by making the substitution uniformly throughout the proof. Once we have a proof of φ , we know automatically that we can prove $s(\varphi)$. There is nothing really to be gained by going through the whole rigamarole all over again. This observation motivates the following addition to the system of rules:

Theorem Substitution Derived Rule (TH). If you have already proved φ from the empty set, you may, at any time in any derivation, write down any substitution instance of φ , again with the empty premiss set.

TH isn’t one of the basic rules of the system. It’s a shortcut rule that we feel we are entitled to use because we know that anything we can derive with the rule we can already derive more laboriously without it. Adopting it is a way of resolving a certain tension. If we want to prove things about the system, it is good to have a streamlined system with as few rules as possible, but if we want to actually want to use the system to show arguments are valid, a more robust system is more practical. Use of derived rules like TH — demonstrably redundant rules that aren’t part of the basic system — let’s us have it both ways. Using TH the proof of DNI is simplified to this:

1	1. P	PI
	2. $(\neg\neg\neg P \rightarrow \neg P)$	TH1
1	3. $\neg\neg P$	MT, 1, 2
	4. $(P \rightarrow \neg\neg P)$	CP, 1, 3

TH1 and TH5 together assure us that $\neg\neg\varphi$ and φ are interderivable. In proofs, we need to be able to go back and forth between them, for example, in the following derivation of the law of Duns Scotus, “ $(\neg P \rightarrow (P \rightarrow Q))$ ”:

1	1. $\neg P$	PI
2	2. P	PI
	3. $(P \rightarrow \neg\neg P)$	TH5
2	4. $\neg\neg P$	MP, 2, 3
2	5. $(\neg Q \rightarrow \neg\neg P)$	CP, 4
1,2	6. Q	MT, 1, 5
1	7. $(P \rightarrow Q)$	CP, 2, 6
TH6	8. $(\neg P \rightarrow (P \rightarrow Q))$	CP, 1, 7

Reductio ad absurdum is the method of proof in which you prove a statement by assuming the negation of the statement and then showing that this assumption leads to a contradiction. The earliest example that I know of is Pythagoras’s proof that the square root of 2 is irrational. Assume, for *reductio ad absurdum*, that $\sqrt{2}$ is rational, so that there are relatively prime positive integers p and q with $\sqrt{2} = p/q$. Multiplying both sides by q and then squaring both

sides gives us $2q^2 = p^2$. This implies that p^2 is even, which in turn implies that p is even, so that there is an r such that $p = 2r$ and $p^2 = 4r^2$. Dividing both sides by 2 gives us $q^2 = 2r^2$. This tells us that q^2 is even, which implies that q is even. But this contradicts the initial assumption that p and q were relatively prime. So we are able to conclude that $\sqrt{2}$ was irrational all along.

To get a formalized version of *reductio ad absurdum* within our deductive system, we prove the law of Clavius, " $((\neg P \rightarrow P) \rightarrow P)$." Once we have this, our method of proving ϕ will be to first assume $\neg \phi$ and go for a contradiction, using the law of Duns Scotus along the way:

1	1. $(\neg P \rightarrow P)$	PI
2	2. $\neg P$	PI
1,2	3. P	MP, 1, 2
	4. $(\neg P \rightarrow (P \rightarrow \neg(\neg P \rightarrow P)))$	TH6
1,2	5. $(P \rightarrow \neg(\neg P \rightarrow P))$	MP, 2, 4
1,2	6. $\neg(\neg P \rightarrow P)$	MP, 3, 5
1	7. $(\neg P \rightarrow \neg(\neg P \rightarrow P))$	CP, 2, 6
1	8. P	MT, 1, 7
TH7	9. $((\neg P \rightarrow P) \rightarrow P)$	CP, 1, 8

As an application, let's prove the theorem, " $((P \rightarrow Q) \rightarrow R) \rightarrow ((P \rightarrow R) \rightarrow R)$." This theorem isn't at all obvious intuitively, but if you work out the truth table you'll see that it's valid. We start out with our basic strategy for proving conditionals, assuming the antecedents and trying to derive the consequents:

1	1. $((P \rightarrow Q) \rightarrow R)$	PI
2	2. $(P \rightarrow R)$	PI
.....		
1,2	n. R	
1	n+1. $((P \rightarrow R) \rightarrow R)$	CP, 2, n
	n+2. $((P \rightarrow Q) \rightarrow R) \rightarrow ((P \rightarrow R) \rightarrow R)$	CP, 1, n+1

Now we're stuck, but we have a new strategy to apply in cases where we're stuck. We assume " $\neg R$ " and try to derive " R ." Then we can discharge the assumption to derive " $(\neg R \rightarrow R)$," from which we can derive " R " by the law of Clavius.

1	1. $((P \rightarrow Q) \rightarrow R)$	PI
2	2. $(P \rightarrow R)$	PI
3	3. $\neg R$	PI
.....		
1,2,3	n-3. R	
1,2	n-2. $(\neg R \rightarrow R)$	CP, 3, n-3
	n-1. $((\neg R \rightarrow R) \rightarrow R)$	TH7
1,2	n. R	MP, n-2, n-1
1	n+1. $((P \rightarrow R) \rightarrow R)$	CP, 2, n
	n+2. $((P \rightarrow Q) \rightarrow R) \rightarrow ((P \rightarrow R) \rightarrow R)$	CP, 1, n+1

Working from the top down, we want to derive “ $\neg P$ ” from lines 2 and 3. The rule of inference

$$\begin{array}{l} (\phi \rightarrow \psi) \\ \neg \psi \\ \therefore \neg \phi \end{array}$$

is sometimes referred to as *modus tollens*, and it’s a perfectly good rule, but it’s not one of the rules of our formal system. Our system only allows *modus tollens* in cases in which the antecedent and consequent are both negated. No problem. We can convert line 2 to “ $(\neg \neg P \rightarrow \neg \neg R)$ ” by applying TH1 and TH5.

1	1.	$((P \rightarrow Q) \rightarrow R)$	PI
2	2.	$(P \rightarrow R)$	PI
3	3.	$\neg R$	PI
4	4.	$\neg \neg P$	PI
	5.	$(\neg \neg P \rightarrow P)$	TH1
4	6.	P	MP, 4, 5
2,4	7.	R	MP, 2, 6
	8.	$(R \rightarrow \neg \neg R)$	TH5
2,4	9.	$\neg \neg R$	MP, 7, 8
2	10.	$(\neg \neg P \rightarrow \neg \neg R)$	CP, 4, 9
2,3	11.	$\neg P$	MT, 3, 10

.....

1,2,3	n-3.	R	
1,2	n-2.	$(\neg R \rightarrow R)$	CP, 3, n-3
	n-1.	$((\neg R \rightarrow R) \rightarrow R)$	TH7
1,2	n.	R	MP, n-2, n-1
1	n+1.	$((P \rightarrow R) \rightarrow R)$	CP, 2, n
	n+2.	$((P \rightarrow Q) \rightarrow R) \rightarrow ((P \rightarrow R) \rightarrow R))$	CP, 1, n+1

An application of the law of Duns Scotus, which permits us to infer “ $(P \rightarrow Q)$ ” from “ $\neg P$,” enables us to complete the proof:

1	1.	$((P \rightarrow Q) \rightarrow R)$	PI
2	2.	$(P \rightarrow R)$	PI
3	3.	$\neg R$	PI
4	4.	$\neg \neg P$	PI
	5.	$(\neg \neg P \rightarrow P)$	TH1
4	6.	P	MP, 4, 5
2,4	7.	R	MP, 2, 6
	8.	$(R \rightarrow \neg \neg R)$	TH5
2,4	9.	$\neg \neg R$	MP, 7, 8
2	10.	$(\neg \neg P \rightarrow \neg \neg R)$	CP, 4, 9
2,3	11.	$\neg P$	MT, 3, 10
	12.	$(\neg P \rightarrow (P \rightarrow Q))$	TH6
2,3	13.	$(P \rightarrow Q)$	MP, 11, 12

1,2,3	14. R	MP, 1, 13
1,2	15. $(\neg R \rightarrow R)$	CP, 3, 14
	16. $((\neg R \rightarrow R) \rightarrow R)$	TH7
1,2	17. R	MP, 15, 16
1	18. $((P \rightarrow R) \rightarrow R)$	CP, 2, 17
TH8	19. $((P \rightarrow Q) \rightarrow R) \rightarrow ((P \rightarrow R) \rightarrow R))$	CP, 1, 18

Our proofs usually look like this. We work from the two ends toward the middle, hoping eventually to connect the two parts. Here's another example, the proof from the empty set of " $((P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q))$ ": Since we're proving conditionals, the first thing we do is to assume the antecedents and try to derive the consequents:

1	1. $(P \rightarrow Q)$	PI
2	2. $(\neg P \rightarrow Q)$	PI
.....		
1,2	n. Q	
1	n+1. $((\neg P \rightarrow Q) \rightarrow Q)$	CP, 2, n
	n+2. $((P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q))$	CP, 1, n+1

Now we work on a *reductio* strategy, assuming " $\neg Q$ " and trying to derive "Q":

1	1. $(P \rightarrow Q)$	PI
2	2. $(\neg P \rightarrow Q)$	PI
3	3. $\neg Q$	PI
.....		
1,2,3	n-3. Q	
1,2	n-2. $(\neg Q \rightarrow Q)$	CP, 3, n-2
	n-1. $((\neg Q \rightarrow Q) \rightarrow Q)$	TH7
1,2	n. Q	MP, n-2, n-1
1	n+1. $((\neg P \rightarrow Q) \rightarrow Q)$	CP, 2, n
	n+2. $((P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q))$	CP, 1, n+1

We want to use lines 2 and 3 to derive "P." Basically, we use MT, but there's an added step of double negation introduction:

1	1. $(P \rightarrow Q)$	PI
2	2. $(\neg P \rightarrow Q)$	PI
3	3. $\neg Q$	PI
4	4. $\neg P$	PI
2,4	5. Q	MP, 2, 4
	6. $(Q \rightarrow \neg\neg Q)$	TH5
2,4	7. $\neg\neg Q$	MP, 5, 6
2	8. $(\neg P \rightarrow \neg\neg Q)$	CP, 4, 7
2,3	9. P	MT, 3, 8

.....
1,2,3 n-3. Q

1,2	n-2. $(\neg Q \rightarrow Q)$	CP, 3, n-2
	n-1. $((\neg Q \rightarrow Q) \rightarrow Q)$	TH7
1,2	n. Q	MP, n-2, n-1
1	n+1. $((\neg P \rightarrow Q) \rightarrow Q)$	CP, 2, n
	n+2. $((P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q))$	CP, 1, n+1

The two ends have finally met, since we can use 1 and 9 to derive n-3 by MP:

1	1. $(P \rightarrow Q)$	PI
2	2. $(\neg P \rightarrow Q)$	PI
3	3. $\neg Q$	PI
4	4. $\neg P$	PI
2,4	5. Q	MP, 2, 4
	6. $(Q \rightarrow \neg\neg Q)$	TH
2,4	7. $\neg\neg Q$	MP, 5, 6
2	8. $(\neg P \rightarrow \neg\neg Q)$	CP, 4, 7
2,3	9. P	MT, 3, 8
1,2,3	10. Q	MP, 1, 9
1,2	11. $(\neg Q \rightarrow Q)$	CP, 3, n-2
	12. $((\neg Q \rightarrow Q) \rightarrow Q)$	TH7
1,2	13. Q	MP, 11, 12
1	14. $((\neg P \rightarrow Q) \rightarrow Q)$	CP, 2, 13
TH9	15. $((P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q))$	CP, 1, 14

It is now time to introduce the rules for the other connectives.

Rule for Definition of Connectives (DC). You may write an instance of any of the following six schemata with the empty premiss set:

- $((\varphi \vee \psi) \rightarrow (\neg \varphi \rightarrow \psi))$
- $((\neg \varphi \rightarrow \psi) \rightarrow (\varphi \vee \psi))$
- $((\varphi \wedge \psi) \rightarrow \neg(\varphi \rightarrow \neg \psi))$
- $(\neg(\varphi \rightarrow \neg \psi) \rightarrow (\varphi \wedge \psi))$
- $((\varphi \leftrightarrow \psi) \rightarrow ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)))$
- $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \rightarrow (\varphi \leftrightarrow \psi))$

Notice that any substitution instance of a sentence we are entitled to write by rule DC is again a sentence are allowed to write by rule DC. It follows that it's legitimate to employ TH with theorems whose use employed TH.

Here are some basic laws for "∨": First, " $(P \rightarrow (P \vee Q))$ ":

1	1. P	PI
	2. $(P \rightarrow (\neg P \rightarrow Q))$	TH6
1	3. $(\neg P \rightarrow Q)$	MP, 1, 2
	4. $((\neg P \rightarrow Q) \rightarrow (P \vee Q))$	DC

1	5. $(P \vee Q)$	MP, 3, 4
TH10	6. $(P \rightarrow (P \vee Q))$	CP, 1, 4

Next, " $(Q \rightarrow (P \vee Q))$ ":

1	1. Q	PI
1	2. $(\neg P \rightarrow Q)$	CP, 1
	3. $((\neg P \rightarrow Q) \rightarrow (P \vee Q))$	DC
1	4. $(P \vee Q)$	MP, 2, 3
TH11	5. $(Q \rightarrow (P \vee Q))$	CP, 1, 4

Finally, the principle of disjunctive syllogism, " $((P \rightarrow R) \rightarrow ((Q \rightarrow R) \rightarrow ((P \vee Q) \rightarrow R)))$ ":

1	1. $(P \rightarrow R)$	PI
2	2. $(Q \rightarrow R)$	PI
3	3. $(P \vee Q)$	PI
	4. $((P \vee Q) \rightarrow (\neg P \rightarrow Q))$	DC
3	5. $(\neg P \rightarrow Q)$	MP, 3, 4
6	6. $\neg P$	PI
3,6	7. Q	MP, 5, 6
2,3,6	8. R	MP, 2, 6
2,3	9. $(\neg P \rightarrow R)$	CP, 6, 8
	10. $((P \rightarrow R) \rightarrow ((\neg P \rightarrow R) \rightarrow R))$	TH9
1	11. $((\neg P \rightarrow R) \rightarrow R)$	MP, 1, 10
1,2,3	12. R	MP, 9, 11
1,2	13. $((P \vee Q) \rightarrow R)$	CP, 3, 12
1	14. $((Q \rightarrow R) \rightarrow ((P \vee Q) \rightarrow R))$	CP, 2, 13
TH12	15. $((P \rightarrow R) \rightarrow ((Q \rightarrow R) \rightarrow ((P \vee Q) \rightarrow R)))$	CP, 1, 14

Now let's prove some laws for " \wedge ": First, " $((P \wedge Q) \rightarrow P)$ ":

1	1. $(P \wedge Q)$	PI
	2. $((P \wedge Q) \rightarrow \neg(P \rightarrow \neg Q))$	DC
1	3. $\neg(P \rightarrow \neg Q)$	MP, 1, 2
4	4. $\neg P$	PI
	5. $(\neg P \rightarrow (P \rightarrow \neg Q))$	TH6
4	6. $(P \rightarrow \neg Q)$	MP, 4, 5
	7. $((P \rightarrow \neg Q) \rightarrow \neg\neg(P \rightarrow \neg Q))$	TH5
4	8. $\neg\neg(P \rightarrow \neg Q)$	MP, 6, 7
	9. $(\neg P \rightarrow \neg\neg(P \rightarrow \neg Q))$	CP, 4, 8
1	10. P	MT, 3, 9
TH13	11. $((P \wedge Q) \rightarrow P)$	CP, 1, 10

Next, " $((P \wedge Q) \rightarrow Q)$ ":

1	1. $(P \wedge Q)$	PI
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	2. $((P \wedge Q) \rightarrow \neg(P \rightarrow \neg Q))$	DC
1	3. $\neg(P \rightarrow \neg Q)$	MP, 1, 2
4	4. $\neg Q$	PI
4	5. $(P \rightarrow \neg Q)$	CP, 4
	6. $((P \rightarrow \neg Q) \rightarrow \neg\neg(P \rightarrow \neg Q))$	TH5
4	7. $\neg\neg(P \rightarrow \neg Q)$	MP, 5, 6
	8. $(\neg Q \rightarrow \neg\neg(P \rightarrow \neg Q))$	CP, 4, 7
1	9. Q	MT, 3, 8
TH14	10. $((P \wedge Q) \rightarrow Q)$	CP, 1, 9

Finally, the conjunction introduction principle, “ $(P \rightarrow (Q \rightarrow (P \wedge Q)))$ ”:

1	1. P	PI
2	2. Q	PI
3	3. $\neg\neg(P \rightarrow \neg Q)$	PI
	4. $(\neg\neg(P \rightarrow \neg Q) \rightarrow (P \rightarrow \neg Q))$	TH1
3	5. $(P \rightarrow \neg Q)$	MP, 3, 4
1,3	6. $\neg Q$	MP, 1, 5
	7. $(\neg Q \rightarrow (Q \rightarrow \neg(P \rightarrow \neg Q)))$	TH6
1,3	8. $(Q \rightarrow \neg(P \rightarrow \neg Q))$	MP, 6, 7
1,2,3	9. $\neg(P \rightarrow \neg Q)$	MP, 2, 8
1,2	10. $(\neg\neg(P \rightarrow \neg Q) \rightarrow \neg(P \rightarrow \neg Q))$	CP, 3, 9
	11. $((\neg\neg(P \rightarrow \neg Q) \rightarrow \neg(P \rightarrow \neg Q)) \rightarrow \neg(P \rightarrow \neg Q))$	TH7
1,2	12. $\neg(P \rightarrow \neg Q)$	MP, 10, 11
	13. $(\neg(P \rightarrow \neg Q) \rightarrow (P \wedge Q))$	DC
1,2	14. $(P \wedge Q)$	MP, 12, 13
1	15. $(Q \rightarrow (P \wedge Q))$	CP, 2, 14
TH15	16. $(P \rightarrow (Q \rightarrow (P \wedge Q)))$	CP, 1, 15

Now that we’ve proved these three theorems, they’re practically the only facts about “ \wedge ” that we’ll ever need. Thus if you’re asked to prove something of the form $(\phi \wedge \psi)$, you’ll do it by first proving ϕ and then ψ , before using the theorem to put the two together. We’ll hardly ever have to go back and make use of the equivalence of $(\phi \wedge \psi)$ with $\neg(\phi \rightarrow \neg\psi)$.

To prove a biconditional, prove the two directions separately. This works, because “ $((P \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow (P \leftrightarrow Q)))$ ” is an SC theorem:

1	1. $(P \rightarrow Q)$	PI
2	2. $(Q \rightarrow P)$	PI
	3. $((P \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow ((P \rightarrow Q) \wedge (Q \rightarrow P))))$	TH15
1	4. $((Q \rightarrow P) \rightarrow ((P \rightarrow Q) \wedge (Q \rightarrow P)))$	MP, 1, 3
1,2	5. $((P \rightarrow Q) \wedge (Q \rightarrow P))$	MP, 2, 4
	6. $((P \rightarrow Q) \wedge (Q \rightarrow P) \rightarrow (P \leftrightarrow Q))$	DC
1,2	7. $(P \leftrightarrow Q)$	MP, 5, 6
1	8. $((Q \rightarrow P) \rightarrow (P \leftrightarrow Q))$	CP, 2, 7
TH16	9. $((P \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow (P \leftrightarrow Q)))$	CP, 1, 8

We also have “ $((P \leftrightarrow Q) \rightarrow (P \rightarrow Q))$ ” and “ $((P \leftrightarrow Q) \rightarrow (Q \rightarrow P))$ ”:

1	1. $(P \leftrightarrow Q)$	PI
	2. $((P \leftrightarrow Q) \rightarrow ((P \rightarrow Q) \wedge (Q \rightarrow P)))$	DC
1	3. $((P \rightarrow Q) \wedge (Q \rightarrow P))$	MP, 1, 2
	4. $((P \rightarrow Q) \wedge (Q \rightarrow P)) \rightarrow (P \rightarrow Q)$	TH13
1	5. $(P \rightarrow Q)$	MP, 3, 4
TH17	6. $((P \leftrightarrow Q) \rightarrow (P \rightarrow Q))$	CP, 1, 5

1	1. $(P \leftrightarrow Q)$	PI
	2. $((P \leftrightarrow Q) \rightarrow ((P \rightarrow Q) \wedge (Q \rightarrow P)))$	DC
1	3. $((P \rightarrow Q) \wedge (Q \rightarrow P))$	MP, 1, 2
	4. $((P \rightarrow Q) \wedge (Q \rightarrow P)) \rightarrow (Q \rightarrow P)$	TH14
1	5. $(Q \rightarrow P)$	MP, 3, 4
TH18	6. $((P \leftrightarrow Q) \rightarrow (Q \rightarrow P))$	CP, 1, 5

To practice proving biconditionals, let’s prove the contraposition principle, “ $((P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P))$,” as an SC theorem. The principle will be helpful, since it let’s us turn a conditional around to a more convenient form. First, the left-to right direction:

1	1. $(P \rightarrow Q)$	PI
2	2. $\neg Q$	PI
3	3. $\neg\neg P$	PI
	4. $(\neg\neg P \rightarrow P)$	TH1
3	5. P	MP, 3, 4
1,3	6. Q	MP, 1, 5
	7. $(Q \rightarrow \neg\neg Q)$	TH5
1,3	8. $\neg\neg Q$	MP, 6, 7
1	9. $(\neg\neg P \rightarrow \neg\neg Q)$	CP, 3, 8
1,2	10. $\neg P$	MT, 2, 9
1	11. $(\neg Q \rightarrow \neg P)$	CP, 2, 10
TH19	12. $((P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P))$	CP, 1, 11

Now, right-to-left:

1	1. $(\neg Q \rightarrow \neg P)$	PI
2	2. P	PI
1,2	3. Q	MT, 1, 2
1	4. $(P \rightarrow Q)$	CP, 2, 3
TH20	5. $((\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q))$	CP, 1, 4

Finally, we put the parts together:

1.	$((P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)) \rightarrow (((\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q)) \rightarrow ((P \rightarrow Q) \leftrightarrow (\neg Q \leftrightarrow \neg P)))$	TH16
2.	$((P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P))$	TH19

	3. $((\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q)) \rightarrow ((P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P))$	MP,1,2
	4. $((\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q))$	TH20
TH21	5. $((P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P))$	MP,3,4

The left-to-right direction is substantially more complicated than its converse because the particular form in which MT is written requires us to perform double negation introduction and elimination. The form of MT isn't entirely arbitrary. If we replaced MT by an the following alternative rule, which is valid and no less natural, we would no longer be able to derive TH1:

If you have derived $\neg \psi$ with premiss set Γ and $(\varphi \rightarrow \psi)$ with premiss set Δ , you may write $\neg \varphi$ with premiss set $\Gamma \cup \Delta$.

As a more complicated example, let's prove one of de Morgan's laws, namely, " $(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$ ":

1	1. $\neg(P \wedge Q)$	PI
2	2. $\neg\neg P$	PI
	3. $(\neg\neg P \rightarrow P)$	TH1
2	4. P	MP, 2, 3
	5. $(P \rightarrow (Q \rightarrow (P \wedge Q)))$	TH15
2	6. $(Q \rightarrow (P \wedge Q))$	MP, 4, 5
	7. $((Q \rightarrow (P \wedge Q)) \rightarrow (\neg(P \wedge Q) \rightarrow \neg Q))$	TH19
2	8. $(\neg(P \wedge Q) \rightarrow \neg Q)$	MP, 6, 7
1,2	9. $\neg Q$	MP, 1, 8
1	10. $(\neg\neg P \rightarrow \neg Q)$	CP, 2, 9
	11. $((\neg\neg P \rightarrow \neg Q) \rightarrow (\neg P \vee \neg Q))$	DC
1	12. $(\neg P \vee \neg Q)$	MP, 10, 11
	13. $(\neg(P \wedge Q) \rightarrow (\neg P \vee \neg Q))$	CP, 1, 12
	14. $((P \wedge Q) \rightarrow P)$	TH13
	15. $((P \wedge Q) \rightarrow P) \rightarrow (\neg P \rightarrow \neg(P \wedge Q))$	TH19
	16. $(\neg P \rightarrow \neg(P \wedge Q))$	MP, 14, 15
	17. $((P \wedge Q) \rightarrow Q)$	TH14
	18. $((P \wedge Q) \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg(P \wedge Q))$	TH19
	19. $(\neg Q \rightarrow \neg(P \wedge Q))$	MP, 17, 18
	20. $((\neg P \rightarrow \neg(P \wedge Q)) \rightarrow ((\neg Q \rightarrow \neg(P \wedge Q)) \rightarrow$ $((\neg P \vee \neg Q) \rightarrow \neg(P \wedge Q))))$	TH12
	21. $(\neg Q \rightarrow \neg(P \wedge Q)) \rightarrow ((\neg P \vee \neg Q) \rightarrow \neg(P \wedge Q))$	MP, 16, 20
	22. $((\neg P \vee \neg Q) \rightarrow \neg(P \wedge Q))$	MP, 19, 21
	23. $((\neg(P \wedge Q) \rightarrow (\neg P \vee \neg Q)) \rightarrow (((\neg P \vee \neg Q) \rightarrow \neg(P \wedge Q))$ $\rightarrow (\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))))$	TH16
	24. $((\neg P \vee \neg Q) \rightarrow \neg(P \wedge Q)) \rightarrow (\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$	MP, 13, 23
TH22	25. $(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$	MP, 22, 24

We've been looking exclusively at argument with the empty premiss set, but we can apply the methods with arbitrary arguments. One reason for focusing on the empty-premiss-set case is that that way we are able to bulk up the collection of SC theorems available for the

application of TH. Now I'd like to look at a case where we start with a English argument, translate it into SC, then prove the English argument valid by deriving the translated conclusion from the translated premisses. Here's the English argument:

Either Preston or Quincy is a member of the Logic Club. If either Quincy or Rudolf is a member, Stuart is not. Unless Stuart is a member, Trumbull is a member and Rudolf is not. But Preston is not a member. Consequently, Quincy and Trumbull are both members.

Now the translation:

$$\begin{aligned} & (P \vee Q) \\ & ((Q \vee R) \rightarrow \neg S) \\ & (\neg S \rightarrow (T \wedge \neg R)) \\ & \neg P \\ & \therefore (Q \wedge T) \end{aligned}$$

Finally, the derivation:

1	1. $(P \vee Q)$	PI
2	2. $((Q \vee R) \rightarrow \neg S)$	PI
3	3. $(\neg S \rightarrow (T \wedge \neg R))$	PI
4	4. $\neg P$	PI
1	5. $(\neg P \rightarrow Q)$	PI
1,4	6. Q	MP, 4, 5
	7. $(Q \rightarrow (Q \vee R))$	TH
1,4	8. $(Q \vee R)$	MP, 6, 7
1,2,4	9. $\neg S$	MP, 2, 8
1,2,3,4	10. $(T \wedge \neg R)$	MP, 3, 9
	11. $((T \wedge \neg R) \rightarrow T)$	TH13
1,2,3,4	12. T	MP, 10, 11
	13. $(Q \rightarrow (T \rightarrow (Q \wedge T)))$	TH15
1,4	14. $(T \rightarrow (Q \wedge T))$	MP, 6, 13
1,2,3,4	15. $(Q \wedge T)$	MP, 12, 14

Basic Rules of Deduction

- PI You may write down any sentence you like if you take the sentence as its own premiss set.
- CP If you have derived ψ with premiss set Γ , you may write $(\phi \rightarrow \psi)$ with premiss set $\Gamma \sim \{\phi\}$.
- MP If you have derived ϕ with premiss set Γ and $(\phi \rightarrow \psi)$ with premiss set Δ , you may write ψ with premiss set $\Gamma \cup \Delta$.
- MT If you have derived ψ with premiss set Γ and $(\neg \phi \rightarrow \neg \psi)$ with premiss set Δ , you may write ϕ with premiss set $\Gamma \cup \Delta$.
- DC You may write an instance of any of the following six schemata with the empty premiss set:
- $((\phi \vee \psi) \rightarrow (\neg \phi \rightarrow \psi))$
 - $((\neg \phi \rightarrow \psi) \rightarrow (\phi \vee \psi))$
 - $((\phi \wedge \psi) \rightarrow \neg(\phi \rightarrow \neg \psi))$
 - $(\neg(\phi \rightarrow \neg \psi) \rightarrow (\phi \wedge \psi))$
 - $((\phi \leftrightarrow \psi) \rightarrow ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)))$
 - $((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) \rightarrow (\phi \leftrightarrow \psi))$

Derived Rule

- TH If you have already proved ϕ from the empty set, you may, at any time in any derivation, write down any substitution instance of ϕ again, with the empty premiss set.

SC Theorems We Have Proved Thus Far

TH1	$(\neg \neg P \rightarrow P)$	Double negation elimination
TH2	$(Q \rightarrow (P \rightarrow Q))$	
TH3	$((P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R)))$	Principle of the syllogism
TH4	$((\neg(P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R)))$	
TH5	$(P \rightarrow \neg \neg P)$	Double negation introduction
TH6	$(\neg P \rightarrow (P \rightarrow Q))$	Law of Duns Scotus
TH7	$((\neg P \rightarrow P) \rightarrow P)$	Law of Clavius
TH8	$((\neg(P \rightarrow Q) \rightarrow R) \rightarrow ((P \rightarrow R) \rightarrow R))$	
TH9	$((P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q))$	
TH10	$(P \rightarrow (P \vee Q))$	A disjunction introduction principle
TH11	$(Q \rightarrow (P \vee Q))$	A disjunction introduction principle
TH12	$((P \rightarrow R) \rightarrow ((Q \rightarrow R) \rightarrow ((P \vee Q) \rightarrow R)))$	Principle of disjunctive syllogism
TH13	$((P \wedge Q) \rightarrow P)$	A conjunction elimination principle
TH14	$((P \wedge Q) \rightarrow Q)$	A conjunction elimination principle
TH15	$(P \rightarrow (Q \rightarrow (P \wedge Q)))$	Conjunction introduction principle
TH16	$((P \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow (P \leftrightarrow Q)))$	
TH17	$((P \leftrightarrow Q) \rightarrow (P \rightarrow Q))$	
TH18	$((P \leftrightarrow Q) \rightarrow (Q \rightarrow P))$	
TH19	$((P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P))$	
TH20	$((\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q))$	
TH21	$((P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P))$	Principle of contraposition
TH22	$(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$	One of de Morgan's laws