Derivations in the Monadic Predicate Calculus

What we want to do now is to obtain a proof procedure of the monadic predicate calculus by adding a few easy rules to the rules we already have for the sentential calculus. Before we begin, let me repeat a point I emphasized when whe introduced the rules of deduction for SC: In an important sense, introducing a proof procedure for the MPC is a giant step backward, since we already have a decision procedure. A decision procedure is vastly better than a proof procedure, for two related reasons: First, a decision procedure gives you a way to show an argument to be invalid, as well as a way to show an argument to be valid. Second, a decision procedure lets you know when it's time to give up trying to show that a given argument is valid, whereas if all you have is a proof procedure, you could find yourself stumbling around forever, searching for a proof that isn't there. The reason why, in spite of these disadvantages, we persist in developing a MPC proof procedure is that we can use the exact same rules when we turn to the full predicate calculus, where no decision procedure is possible.

The basic structure of our deductions will be unchanged from the SC. A proof will be a numbered sequence of sentences, each with an associated premiss set, written down according to rules that ensure that each of the numbered sentences is a logical consequence of its premiss set. The basic rules of SC have this feature, so we make take them over *verbatim*:

(PI) You may write down any sentence you like, taking the unit set of that sentence as your premiss set.

(CP) If you have obtained ψ with premiss set Γ, you may write $(\varphi \rightarrow \psi)$ with premiss set $\Gamma \sim \{\varphi\}$.

(MP) If you have written φ with premiss set Γ and $(\varphi \rightarrow \psi)$ with premiss set Δ, you may write ψ with premiss set $\Gamma \cup \Delta$.

(MT) If you have written ψ with premiss set Γ and $(\neg \varphi \rightarrow \neg \psi)$ with premiss set Δ, you may write φ with premiss set $\Gamma \cup \Delta$.

(DC) You may write down any instance of any of the following schemata with the empty set of premisses:
$((\varphi \lor \psi) \rightarrow (\neg \varphi \rightarrow \psi))$
$((\neg \varphi \rightarrow \psi) \rightarrow (\varphi \lor \psi))$
$((\varphi \land \psi) \rightarrow \neg (\varphi \rightarrow \neg \psi))$
$(\neg (\varphi \rightarrow \neg \psi) \rightarrow (\varphi \land \psi))$
$((\varphi \leftrightarrow \psi) \rightarrow ((\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi))$
$(((\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)) \rightarrow (\varphi \leftrightarrow \psi))$

We can also take over the derived rule TH:

(TH) You may write down any MPC sentence that is a substitution instance of a SC theorem with the empty premiss set.

Here a "substitution instance" of an SC sentence will be an MPC sentence that can be obtained from the SC sentence by substituting MPC sentences from atomic SC sentences, making sure that different occurrences of the same atomic sentence can be replaced in the same way. We could, if we wanted to, after we'd introduced all the basic MPC rules, formulate an extension of rule TH where you start with a theorem of MPC (that is, an MPC sentence that you've derived from the empty set) and get a new new MPC theorem by uniformly substituting a formula for each predicate that occurs in the original theorem. But the further rule wouldn't be very useful, so that introducing it would be more trouble than it's worth.

Rather than reintroducing the derived rules SE and BI, we incorporate them into a very powerful derived rule that assimilates the entire sentential calculus:

> **Tautological consequence rule (TC).** If you have written down sentences ψ_1, ψ_2,..., ψ_n in a derivation, and φ is a tautological consequence of $\{\psi_1,\psi_2,...,\psi_n\}$, then you may write down sentence ψ, taking the premiss set to be the union of the premiss sets of the ψ_is. In particular, if φ is a tautology, we can write φ with the empty premiss set.

We know from the completeness theorem for the sentential calculus that, if φ is a tautological consequence of $\{\psi_1,\psi_2,...,\psi_n\}$, then the conditional $(\psi_1 \rightarrow (\psi_2 \rightarrow ... \rightarrow (\psi_n \rightarrow \varphi)...))$ can be derived from the empty set by means of rules PI, CP, MP, MT, and DC. By first writing the derivations of the ψ_is, one after another, followed by the derivation from the empty set of $(\psi_1 \rightarrow (\psi_2 \rightarrow ... \rightarrow (\psi_n \rightarrow \varphi)...))$, followed by n steps of MP, and finishing up with φ, we can get a derivation of φ with the union of the premiss sets of the ψ_is as its premiss set without making use of TC. This is what entitles TC to its humble title of "derived rule."

If, in rule TC, you replaced the words "tautological consequence" and "a tautology" with "logical consequence" and "valid," you would still get a rule that had the property that anything you wrote down on the basis of the rule would be a logical consequence of its premiss set, but it would be a pretty silly rule, since you could only apply the rule if you already knew how to recognize when one sentence is a logical consequence of some others, and the whole purpose of introducing the system of rules is to help us recognize when one sentence is a logical consequence of some others. Rule TC isn't silly in this way, because we already have an efficient method for testing whether φ is a tautological consequence of $\{\psi_1, \psi_2.,,, \psi_n\}$, namely the search-for-counterexamples technique, so we can readily test whether a purported applicationo of TC is legitimate. (To be sure, we also have a method of testing whether the MPC sentence ψ is a logical consequence of $\{\psi_1,\psi_2,...,\psi_n\}$, but it's highly impractical, and moreover, we won't have any such method when we turn to full predicate calculus.)

The reason for introducing TC is that we've already learned about SC and now we want to learn what's new in MPC, and rule TC lets us focus our attention where our interest is. As an illustration, using "Wx" for "x went up the hill," "a" for "Jack," "b" for "Jill," and "c" for "Clarissa," let's give a derivation of "Either Jill or Clarissa went up the hill" ["(Wb ∨ Wc)"] from

"Either Jack or Jill went up the hill" ["(Wa ∨ Wb)"] and "If Jack went up the hill, so did Clarissa" ["(Wa → Wc)"]:

1	1. (Wa ∨ Wb)	PI
2	2. (Wa → Wc)	PI
1,2	3. (Wb ∨ Wc)	TC, 1, 2

We can verify that line 3 is truly a tautological consequence of lines 1 and 2 by the search-for-counterexample method, putting a "1" under the main connective of each premiss, and a "0" under the main connective of the conclusion:

(Wa ∨ Wb)	(Wa → Wc)	∴(Wb ∨ Wc)
~~1 0~~	~~0 1 0~~	~~0 0 0~~

Without rule TC, the derivation would be longer:

1	1. (Wa ∨ Wb)	PI
2	2. (Wa → Wc)	PI
	3. ((Wa ∨ Wb) → (¬ Wa → Wb)	DC
1	4. (¬ Wa → Wb)	MP 51, 3
5	5. ¬ Wa	PI
1,5	6. Wb	MP 4, 5
	7. (Wb → ¬ ¬ Wb)	TH5
1,5	8. ¬ ¬ Wb	MP 6, 7
1	9. (¬ Wa → ¬ ¬ Wb)	CP 5, 8
10	10. ¬ Wb	PI
1,10	11. Wa	MT 9, 10
1,2,10	12. Wc	MP 2, 11
1,2	13. (¬ Wb → Wc)	CP 10, 12
	14. ((¬ Wb → Wc) → (Wb ∨ Wc))	DC
1,2	15. (Wb ∨ Wc)	MP 13, 14

So far, our derivations have just been SC derivations in a different notation. Now we're going to learn some rules that describe the operation of the quantifiers. Once again, the rules we'll use are taken over from Mates' *Elementary Logic*:

> **Universal specification rule (US).** If you've derived (∀x)φ, you may derive $\varphi^{x}/_{c}$ with the same premiss set, for any constant c.

For example, we derive "Ms" ("Socrates is mortal") from "Gs" ("Socrates is a Greek") and "(∀x)(Gx → Mx)" ("All Greeks are mortal"):

1	1. Gs	PI
2	2. $(\forall x)(Gx \rightarrow Mx)$	PI
2	3. $(Gs \rightarrow Ms)$	US,2
1,2	4. Ms	MP 1,3

It's clear that rule US is logical-consequence preserving, because $\varphi^{x}/_{c}$ is a logical consequence of $(\forall x)\varphi$. If $(\forall x)\varphi$ is true under $\mathfrak{A}$, every member of $|\mathfrak{A}|$ satisfies φ under $\mathfrak{A}$. So, in particular, $\mathfrak{A}(c)$ satisfies φ under $\mathfrak{A}$, so that, by the Substitution Principle, $\varphi^{x}/_{c}$ is true under $\mathfrak{A}$.

> **Universal generalization rule (UG).** If you've derived $\varphi^{x}/_{c}$ from Γ and if the constant c doesn't appear in φ or in any of the sentences in Γ, you may derive $(\forall x)\varphi$ with premiss set Γ.

For example, we derive "$(\forall x)(Gx \rightarrow Mx)$" ("All Greeks are mortal") from "$(\forall x)(Gx \rightarrow Hx)$" ("All Greeks are human beings") and "$(\forall x)(Hx \rightarrow Mx)$" ("All human beings are mortal"):

1	1. $(\forall x)(Gx \rightarrow Hx)$	PI
2	2. $(\forall x)(Hx \rightarrow Mx)$	PI
3	3. Ga	PI
1	4. $(Ga \rightarrow Ha)$	US,1
1,3	5. Ha	MP 3,4
2	6. $(Ha \rightarrow Ma)$	US,2
1,2,3	7. Ma	MP 5,6
1,2	8. $(Ga \rightarrow Ma)$	CP,3,7
1,2	9. $(\forall x)(Gx \rightarrow Mx)$	UG,8

Here, line 8 is gotten from "$(Gx \rightarrow Mx)$," by substituting "a" for free occurrences of "x," while line nine is gotten from the same formula by prefixing the universal quantifier "$(\forall x)$." Since the constant "a" doesn't appear in the formula "$(Gx \rightarrow Mx)$" and it doesn't appear in the premisses of line 8, we can derive line 9 from line 8 by rule UG.

The idea behind rule UG is that, if you have derived $\varphi^{x}/_{c}$ from Γ, where c doesn't appear in Γ or in φ, then the reason you know $\varphi^{x}/_{c}$ is true if all the members of Γ are true can't have anything special to do with the particular individual named by c, because you don't know anything about the particular individual named by c; that individual isn't even mentioned in the premiss set. Whatever reasons you have for believing that the individual named by c satisfies φ are reasons that would apply just as well to any other element of the universe. So every member of the universe satisfies φ. So $(\forall x)\varphi$ is true.

More formally, we show that rule UG is logical-consequence preserving, as follows: Suppose that $\varphi^{x}/_{c}$ is a logical consequence of Γ and that the constant c doesn't appear in φ or in any of the members of Γ. We want to see that $(\forall x)\varphi$ is a logical consequence of Γ.

Take an interpretation $\mathfrak{A}$ under which all the members of Γ are true. We want to see that $(\forall x)\varphi$ is true under $\mathfrak{A}$. Take a member a of $|\mathfrak{A}|$. We want to show that a satisfies φ under $\mathfrak{A}$. Since a was chosen arbitrarily, this will tell us that every member of $|\mathfrak{A}|$ satisfies φ under $\mathfrak{A}$, so that $(\forall x)\varphi$ is true under $\mathfrak{A}$.

Let $\mathfrak{B}$ be an interpretation which is just like $\mathfrak{A}$ except that $\mathfrak{B}(c) = a$. It follows from the Locality Principle that all the members of Γ are true under $\mathfrak{B}$. Hence $\varphi^x/_c$ is true under $\mathfrak{B}$. It follows by the Substitution Principle that $\mathfrak{B}(c)$, which is a, satisfies φ under $\mathfrak{B}$. Using the Locality Principle again, we know that a satisfies φ under $\mathfrak{A}$, which is what we wanted to show.

As another example, let's derive "$(\forall x)(Cx \vee Ax)$" ("Everyone is either a child or an adult") from "$(\forall x)(Cx \vee (Mx \vee Wx))$," "$(\forall x)(Mx \rightarrow Ax)$," and "$(\forall x)(Wx \rightarrow Ax)$"; we want to fill in the blank in this:

1	1. $(\forall x)(Cx \vee (Mx \vee Wx))$	PI
2	2. $(\forall x)(Mx \rightarrow Ax)$	PI
3	3. $(\forall x)(Wx \rightarrow Ax)$	PI
1,2,3	$(\forall x)(Cx \vee Ax)$	

The sentence we're trying to prove is universal. There is a general strategy for proving universal sentences: To prove $(\forall x)\varphi$, pick a new constant c that doesn't appear anywhere else in the proof, and try to prove $\varphi^x/_c$ with the same premisses; then use UG. Thus we try to prove "$(Cc \vee Ac)$":

1	1. $(\forall x)(Cx \vee (Mx \vee Wx))$	PI
2	2. $(\forall x)(Mx \rightarrow Ax)$	PI
3	3. $(\forall x)(Wx \rightarrow Ax)$	PI
1,2,3	$(Cc \vee Ac)$	
1,2,3	$(\forall x)(Cx \vee Ax)$	UG,

We're now trying to prove something about the individual named by "c." What we know is a bunch of universal statements. From those universal statements, we are trying to derive conclusions about the individual named by "c," and the obvious way to do this is to use rule US:

1	1. $(\forall x)(Cx \vee (Mx \vee Wx))$	PI
2	2. $(\forall x)(Mx \rightarrow Ax)$	PI
3	3. $(\forall x)(Wx \rightarrow Ax)$	PI

4	4. (Cc ∨ (Mc ∨ Wc))	US,1
5	5. (Mc → Ac)	US,2
6	6. (Wc → Ac)	US,3
1,2,3	(Cc ∨ Ac)	
1,2,3	(∀x)(Cx ∨ Ax)	UG,

Now the sentence we're trying to prove is a disjunction. We don't have any general strategy for proving a disjunction, but we do have a strategy for proving conditionals: Assume the antecedent and try to prove the consequent. So what we want to do is to convert the sentence we are trying to prove, "(Cc ∨ Ac)" into the tautologically equivalent conditional "(¬Cc → Ac)," then to assume "¬Cc" as a premiss and try to derive "Ac":

1	1. (∀x)(Cx ∨ (Mx ∨ Wx))	PI
2	2. (∀x)(Mx → Ax)	PI
3	3. (∀x)(Wx → Ax)	PI
4	4. (Cc ∨ (Mc ∨ Wc))	US,1
5	5. (Mc → Ac)	US,2
6	6. (Wc → Ac)	US,3
7	7. ¬Cc	PI
1,2,3,7	Ac	
1,2,3	(¬Cc → Ac)	CP,7,
1,2,3	(Cc ∨ Ac)	TC,
1,2,3	(∀x)(Cx ∨ Ax)	UG,

Lines 4 and 7 together tautologically imply "(Mx ∨ Wc)"; one way to see this is to rewrite line 4 as "(¬Cc → (Mc ∨ Wc))" and to apply *modus ponens*:

1	1. (∀x)(Cx ∨ (Mx ∨ Wx))	PI
2	2. (∀x)(Mx → Ax)	PI
3	3. (∀x)(Wx → Ax)	PI
1	4. (Cc ∨ (Mc ∨ Wc))	US,1
2	5. (Mc → Ac)	US,2
3	6. (Wc → Ac)	US,3
7	7. ¬Cc	PI
1	8. (¬ Cc → (Mc ∨ Wc))	TC 4
1,7	9. (Mc ∨ Wc)	MP 4,8
1,2,3,7	Ac	
1,2,3	(¬Cc → Ac)	CP,7,
1,2,3	(Cc ∨ Ac)	TC,
1,2,3	(∀x)(Cx ∨ Ax)	UG,

Lines 9, 5, and 6 tautologically imply "Ac," which is what we want:

1	1. $(\forall x)(Cx \vee (Mx \vee Wx))$	PI
2	2. $(\forall x)(Mx \rightarrow Ax)$	PI
3	3. $(\forall x)(Wx \rightarrow Ax)$	PI
1	4. $(Cc \vee (Mc \vee Wc))$	US,1
2	5. $(Mc \rightarrow Ac)$	US,2
3	6. $(Wc \rightarrow Ac)$	US,3
7	7. $\neg Cc$	PI
1	8. $(\neg Cc \rightarrow (Mc \vee Wc))$	TC 4
1,7	9. $(Mc \vee Wc)$	MP 4,8
1,2,3,7	10. Ac	TC,5,6,8
1,2,3	11. $(\neg Cc \rightarrow Ac)$	CP,7,10
1,2,3	12. $(Cc \vee Ac)$	TC,11
1,2,3	13. $(\forall x)(Cx \vee Ax)$	UG,11

The last move we made in joining the two ends of our proof was an instance of a general strategy to use when one of the things you know is a disjunction. If you have $(\varphi \vee \psi)$ and you're trying to prove θ, try to prove $(\varphi \rightarrow \theta)$ and $(\psi \rightarrow \theta)$. Then you can put the pieces together by rule TC.

Whenever you derive a sentence φ with the premiss set Γ, you'll know that φ is a logical consequence of Γ. In particular, if you derive φ from the empty set of premisses, you can conclude that φ is a logical consequence of the empty set, that is, you can conclude that φ is valid.

As an example, let's derive "$((\forall x)(Fx \wedge Gx) \leftrightarrow ((\forall x)Fx \wedge (\forall x)Gx))$" from the empty set. Our familiar strategy of proving a biconditioal by proving its two directions separately still serves, although the rule we apply is now called "TC"; BI is just a special case of TC. So we want to fill in the blanks in this:

$((\forall x)(Fx \wedge Gx) \rightarrow ((\forall x)Fx \wedge (\forall x)Gx))$	
$(((\forall x)Fx \wedge (\forall x)Gx) \rightarrow (\forall x)(Fx \wedge Gx))$	
$((\forall x)(Fx \wedge Gx) \leftrightarrow ((\forall x)Fx \wedge (\forall x)Gx))$	TC,

To fill in the first blank, we assume the antecedent and try to derive the consequent:

1	1. $(\forall x)(Fx \wedge Gx)$	PI

1	$((\forall x)Fx \wedge (\forall x)Gx)$	
	$((\forall x)(Fx \wedge Gx) \rightarrow ((\forall x)Fx \wedge (\forall x)Gx))$	CP,1
	$(((\forall x)Fx \wedge (\forall x)Gx) \rightarrow (\forall x)(Fx \wedge Gx))$	
	$((\forall x)(Fx \wedge Gx) \leftrightarrow ((\forall x)Fx \wedge (\forall x)Gx))$	TC

To prove a conjunction, try to prove both conjuncts:

1	1. $(\forall x)(Fx \wedge Gx)$	PI
1	$(\forall x)Fx$	
1	$(\forall x)Gx$	
1	$((\forall x)Fx \wedge (\forall x)Gx)$	TC
	$((\forall x)(Fx \wedge Gx) \rightarrow ((\forall x)Fx \wedge (\forall x)Gx))$	CP
	$(((\forall x)Fx \wedge (\forall x)Gx) \rightarrow (\forall x)(Fx \wedge Gx))$	
	$((\forall x)(Fx \wedge Gx) \rightarrow ((\forall x)Fx \wedge (\forall x)Gx))$	TC

To prove a universal statement, prove an instance of it with a new constant. We use this strategy to prove both "(∀x)Fx" and "(∀x)Gx":

1	1. (∀x)(Fx ∧ Gx)	PI
1	2. (Fa ∧ Ga)	US,1
1	3. Fa	TC,2
1	4. (∀x)Fx	UG,3
1	5. (Fb ∧ Gb)	US,1
1	6. Gb	TC,5
1	7. (∀x)Gx	UG,6
1	8. ((∀x)Fx ∧ (∀x)Gx)	TC,4,7
	9. ((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx))	CP,1,8
	(((∀x)Fx ∧ (∀x)Gx) → (∀x)(Fx ∧ Gx))	
	((∀x)(Fx ∧ Gx) ↔ ((∀x)Fx ∧ (∀x)Gx))	TC,

To fill in the remaining blank, we assume the antecedent and try to derive the consequent.

1	1. (∀x)(Fx ∧ Gx)	PI
1	2. (Fa ∧ Ga)	US,1
1	3. Fa	TC,2
1	4. (∀x)Fx	UG,3
1	5. (Fb ∧ Gb)	US,1
1	6. Gb	TC,5
1	7. (∀x)Gx	UG,6
1	8. ((∀x)Fx ∧ (∀x)Gx)	TC,4,7
	9. ((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx))	CP,1,8
10	10. ((∀x)Fx ∧ (∀x)Gx)	PI
10	(∀x)(Fx ∧ Gx)	
	(((∀x)Fx ∧ (∀x)Gx) → (∀x)(Fx ∧ Gx))	CP,10,
	((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx))	TC

What we have to work from is a conjunction, "((∀x)Fx ∧ (∀x)Gx)"; we simplify this by writing its two conjuncts on separate lines. What we're trying to prove is a universal sentence, "(∀x)(Fx ∧ Gx)," which we prove by first proving "(Fc ∧ Gc), intending to apply rule UG:

1	1. (∀x)(Fx ∧ Gx)	PI

1	2. (Fa ∧ Ga)	US,1
1	3. Fa	TC,2
1	4. (∀x)Fx	UG,3
1	5. (Fb ∧ Gb)	US,1
1	6. Gb	TC,5
1	7. (∀x)Gx	UG,6
1	8. ((∀x)Fx ∧ (∀x)Gx)	TC,4,7
	9. ((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx))	CP,1,8
10	10. ((∀x)Fx ∧ (∀x)Gx)	PI
10	11. (∀x)Fx	TC,10
10	12. (∀x)Gx	TC,10
10	13. Fc	US,11
10	14. Gc	US,12
10	15. (Fc ∧ Gc)	TC,13,14
10	16. (∀x)(Fx ∧ Gx)	UG,15
	17. (((∀x)Fx ∧ (∀x)Gx) → (∀x)(Fx ∧ Gx))	CP,10,16
	18. ((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx))	TC,9,17

As another example, let's derive "(Fa → (∀x)(Gx → Fa))" from the empty set:

1	1. Fa	PI
1	2. (Gb → Fa)	TC,1
1	3. (∀x)(Gx → Fa)	UG,2
	4. (Fa → (∀x)(Gx → Fa))	CP,1,3

Let's derive "((∀x)(Fx → Gx) → ((∀x)Fx → (∀x)Gx))" from the empty set:

1	1. (∀x)(Fx → Gx)	PI
2	2. (∀x)Fx	PI
1	3. (Fa → Ga)	US,1
2	4. Fa	US,2
1,2	5. Ga	TC,3,4
1,2	6. (∀x)Gx	UG,5
1	7. ((∀x)Fx → (∀x)Gx)	CP,2,6
	8. ((∀x)(Fx → Gx) → ((∀x)Fx → (∀x)Gx))	CP,1,7

Remember the restriction on rule UG. If you use rule UG to derive $(\forall x)\varphi$ from $\varphi^x/_c$, the constant c shouldn't appear either in φ or in any members of the premiss set of $\varphi^x/_c$. Otherwise, you'll get poppycock, like the following bad derivation from the empty set of "(Ws → (∀x)Wx)" ("If Socrates is wise, everyone is wise"):

1	1. Ws	PI	
1	2. $(\forall x)Wx$		Bad use of UG,1
	3. $(Ws \rightarrow (\forall x)Wx)$	CP,1,2	

We now have all the rules we need for dealing with the universal quantifier, and it's time to turn to the existential quantifier. We'll learn two rules, one easy and the other more complicated:

> **Existential generalization rule (EG).** If you have written $\varphi^x/_c$, for any constant c, you may write $(\exists x)\varphi$ with the same premiss set.

It's easy to see that this new rule is logical-consequence preserving, since $(\exists x)\varphi$ is a logical consequence of $\varphi^x/_c$.

As an illustration, let's derive "$(\exists x)(Gx \wedge Bx)$" ("Some Greeks have beards") from "Gs" ("Socrates is a Greek"), "Ps" ("Socrates is a philosopher"), and "$(\forall x)(Px \rightarrow Bx)$" ("All philosophers have beards"):

1	1. Gs	PI
2	2. Ps	PI
3	3. $(\forall x)(Px \rightarrow Bx)$	PI
3	4. $(Ps \rightarrow Bs)$	US 3
2,3	5. Bs	MP 2,4
1,2,3	6. $(Gs \wedge Ws)$	TC,1,5
1,2,3	7. $(\exists x)(Gx \wedge Wx)$	EG,6

We now rules that let us derive the consequences of a universal premiss (US), that let us derive a universal conclusion (UG), and that let us derive an existential conclusion (EG). What we lack is a rule for deriving the consequences of an existential premiss. The motivation for the new rule is this: Suppose we know that there is something that has property P. Let's pick such a thing and call it "c," If we are able to derive a conclusion that doesn't say anything specific about the thing named by "c," and if deriving the conclusion we don't use anything about the thing we have temporarily named "c" other than that it has property P, then the conclusion we've reached doesn't really depend on the assumption that the thing named by "c" has property P; it really only depends on the the thesis that there is something with property P. We formalize this argumentative strategy in the following rule:

> **Existential specification (ES).** Suppose that you have derived $(\exists x)\varphi$ with premiss set Δ and that you have derived ψ with premiss set $\Gamma \cup \{\varphi^x/_c\}$, for some constant c. Suppose further that the constant c does not appear in φ, in ψ, or in any member of Γ. Then you may derive ψ with premiss set $\Delta \cup \Gamma$.

Here's an example: We derive "(∃x)(Mx ∧ Fx)" ("Some mammals fly") from "(∃x)(Sx ∧ Fx)" ("Some squirrels fly") and "(∀x)(Sx → Mx)" ("All squirrels are mammals"):

1	1. (∃x)(Sx ∧ Fx)	PI
2	2. (∀x)(Sx → Mx)	PI
3	3. (Sa ∧ Fa)	PI (for ES)
3	4. Sa	TC,3
3	5. Fa	TC,3
2	6. (Sa → Ma)	US,2
2,3	7. Ma	MP 4,6
2,3	8. (Ma ∧ Fa)	TC,5,7
2,3	9. (∃x)(Mx ∧ Fx)	EG,8
1,2	10. (∃x)(Mx ∧ Fx)	ES,1,3,9

Here we were trying to prove "(∃x)(Mx ∧ Fx)," and one of the premisses we had available to use was the existential sentence "(∃x)(Sx ∧ Fx)." The way we used this existential premiss was to take an instance of the existential premiss as a new premiss at line 3. Then, at line 9, we used this new premiss to prove "(∃x)(Mx ∧ Fx)." At line 10, we used rule ES to upgrade our premisses, replacing "(Sa ∧ Fa)" in the premiss set of "(∃x)(Mx ∧ Fx)" with the premiss set of "(∃x)(Sx ∧ Fx)."

We know that some squirrels fly. Take a flying squirrel; call him "a." From the assumption that a is a flying squirrel, we are able to conclude that some mammals fly. So this conclusion must already be entailed by "Some squirrels fly."

Notice that what rule ES permits you to do is merely to repeat a sentence you've already written earlier. What's changed is your premiss set. Notice also that an application of ES is always paired which an application of PI. PI occurs up to, ES at the bottom.

As another example, let's derive "(∃x)(Fx ∧ Gx)" from "(∃x)Fx" and "(∀x)Gx":

1	1. (∃x)Fx	PI
2	2. (∀x)Gx	PI
3	3. Fa	PI (for ES)
2	4. Ga	US,2
2,3	5. (Fa ∧ Ga)	TC,3,4
2.3	6. (∃x)(Fx ∧ Gx)	EG,5
1,2	7. (∃x)(Fx ∧ Gx)	ES,1,3,6

We had "(∃x)Fx" as a premiss. We made use of it by taking "Fa" as a premiss, then using rule ES to upgrade the premiss set of our conclusion.

There are two kinds of premisses: premisses we regard as hypotheses of whatever argument we're given, and premisses we introduce during the course of an argument on which we don't want our final conclusion to depend. There's no point in introducing a premiss of the latter sort unless you have some way of removing it from the premiss set of your eventual conclusion, and we have two ways of removing something from the premiss set, CP and ES. To help keep track of where we're going, whenever we introduce a new premiss that isn't intended to be a premiss of the final argument, I've take to marking it as either "for CP" or "for ES." Don't introduce a premiss unless you have a plan for getting rid of it.

Now let's derive "(∃x)(Fx ∧ Gx)" from the premisses "(∃x)(Ax ∧ Bx)," "(∀x)(Ax → Fx)," and "(∀x)(Bx → Gx)":

1	1. (∃x)(Ax ∧ Bx)	PI
2	2. (∀x)(Ax → Fx)	PI
3	3. (∀x)(Bx → Gx)	PI
4	4. (Aa ∧ Ba)	PI (for ES)
4	5. Aa	TC,4
4	6. Ba	TC,4
2	7. (Aa → Fa)	US,2
2,4	8. Fa	MP,5,7
3	9. (Ba → Ga)	US,3
3,4	10. Ga	MP,6,9
2,3,4	11. (Fa ∧ Ga)	TC,8,10
2,3,4	12. (∃x)(Fx ∧ Gx)	EG,11
1,2,3	13. (∃x)(Fx ∧ Gx)	ES,1,4,12

Why does ES work? The reasoning that shows that ES is logical-consequence preserving is similar to the reasoning behind UG. Suppose that (∃x)φ is a logical consequence of Δ and that ψ is a logical consequence of Γ ∪ $\varphi^x/_c$, where c doesn't appear in φ or ψ or in any of the elements of Γ. Let 𝔄 be an interpretation under which every member of Γ ∪ Δ is true. Then (∃x)φ is true under 𝔄, so there is an element s of |𝔄| that satisfies φ under 𝔄. Let 𝔅 be an interpretation that's like 𝔄 except that 𝔅(c) = s. The Locality Principle tells us that s satisfies φ under 𝔅, and so the Substitution Principle assures us that $\varphi^x/_c$ is true under 𝔅. The locality principle assures us that all the members of Γ are true under 𝔅, so we can conclude that ψ is true under 𝔅. By the Locality Principle, ψ is true under 𝔄 as well.

We'll introduce one last rule. It isn't strictly necessary — it's only a derived rule — but it's easy and useful:

Quantifier exchange rule (QE).
From ¬(∀x)¬φ, you may infer (∃x)φ with the same premiss set, and *vice versa.*
From (∀x)¬φ, you may infer ¬(∃x)φ with the same premiss set, and *vice versa.*
From ¬(∀x)φ, you may infer (∃x)¬φ with the same premiss set, and *vice versa.*
From (∀x)φ, you may infer ¬(∃x)¬φ with the same premiss set, and *vice versa.*

To see that the first of these four rules is consequence-preserving, we'll verify that, even without the new rule, we can derive $(\exists x)\varphi$ from $\{\neg (\forall x) \neg \varphi\}$, and derive $\neg (\forall x)\neg \varphi$ from $(\exists x)\varphi$:

1	1. $\neg (\forall x)\neg \varphi$	PI
2	2. $\neg (\exists x)\varphi$	PI (for CP)
3	3. $\varphi^{x}/_{c}$	PI (for CP)
3	4. $(\exists x)\varphi$	EG 3
	5. $(\varphi^{x}/_{c} \rightarrow (\exists x)\varphi)$	CP 3, 4
2	6. $\neg \varphi^{x}/_{c}$	TC 2, 5
2	7. $(\forall x) \neg \varphi$	UG 6
	8. $(\neg(\exists x)\varphi \rightarrow (\forall x)\neg\varphi)$	CP 2, 7
1	9. $(\exists x)\varphi$	TC 1, 8

1	1. $(\exists x)\varphi$	PI
2	2. $\varphi^{x}/_{a}$	PI (for ES)
3	3. $(\forall x)\neg \varphi$	PI (for CP)
3	4. $\neg \varphi^{x}/_{a}$	US 3
	5. $((\forall x)\neg \varphi \rightarrow \neg \varphi^{x}/_{a})$	CP 3, 4
2	6. $\neg (\forall x)\neg \varphi$	TC 2, 5
1	7. $\neg (\forall x)\neg \varphi$	ES 1, 2, 6

The derivations for the other three rules are similar, and we won't go through them here.

The new rule is useful, because, although we already had techniques for dealing with sentences that began with quantifiers, up until now we didn't have any direct way to deal with sentences that begin with negated quantiers. As an illustration, let's derive "$\neg(\exists x)(Mx \wedge Wx)$" ("No one is both a man and a woman") from "$(\forall x)(Mx \rightarrow Bx)$" ("All men have beards") and "$\neg(\exists x)(Wx \wedge Bx)$" ("No women have beards"):

1	1. $(\forall x)(Mx \rightarrow Bx)$	PI
2	2. $\neg(\exists x)(Wx \wedge Bx)$	PI
2	3. $(\forall x)\neg(Wx \wedge Bx)$	QE,2
1	4. $(Ma \rightarrow Ba)$	US,1
2	5. $\neg(Wa \wedge Ba)$	US,3
6	6. Ma	PI (for CP)
1,6	7. Ba	MP 4, 6
1,2,6	8. $\neg Wa$	TC 5, 7
1,2	9. $(Ma \rightarrow \neg Wa)$	CP 6, 8
1,2	10. $\neg (Ma \wedge Wa)$	TC 9
1,2	11. $(\forall x)\neg (Mx \wedge Wx)$	UG 10
1,2	12. $\neg(\exists x)(Mx \wedge Wx)$	QE 11

Now let's derive "$\neg(\forall x)(Mx \rightarrow Lx)$" ("Not every mammal bears live young") from "$\neg(\exists x)(Kx \wedge Lx)$" ("No kangaroos bear live young"), "$\neg(\exists x)(Kx \wedge \neg Mx)$" ("There aren't any kangaroos that aren't mammals"), and "$(\exists x)Kx$" ("There are kangaroos").

1	1. $\neg (\exists x)(Kx \wedge Lx)$	PI
2	2. $\neg(\exists x)(Kx \wedge \neg Mx)$	PI
3	3. $(\exists x)Kx$	PI
4	4. Ka	PI (for ES)
1	5. $(\forall x)\neg (Kx \wedge Lx)$	QE 1
1	6. $\neg (Ka \wedge La)$	US 5
1,4	7. $\neg La$	TC 4, 6
2	8. $(\forall x)\neg (Kx \wedge \neg Mx)$	QE 2
2	9. $\neg (Ka \wedge \neg Ma)$	US 8
2,4	10. Ma	TC 4, 9
1,2,4	11. $(Ma \wedge \neg La)$	TC 7, 10
1,2,4	12. $\neg (Ma \rightarrow La)$	TC 11
1,2,4	13. $(\exists x)\neg (Mx \rightarrow Lx)$	EG 12
1,2,4	14. $\neg (\forall x)(Mx \rightarrow Lx)$	QE 13
1,2,3	15. $\neg (\forall x)(Mx \rightarrow Lx)$	ES 3, 4, 14

Now that we have the rules, let's talk about some strategies for applying the rules. These strategies aren't part of the rules; they're techniques for using the rules efficiently. To have a correct derivation, you have to follow the rules, but you don't have to follow the strategies if you don't want to. The strategies don't always work, but they usually do.

The basic plan is always to work from two ends toward the middle. Thus, at each stage of a derivation, there will be a set of sentences you are assumed to know, and there are one or more sentences you are trying to prove. The strategies are techniques for breaking up the sentences you are working with into sentences that are simpler and therefore, hopefully, easier to work with.

There are two groups of strategies, bottom up strategies simplifying the sentence you're trying to prove, and top down strategies for simplifying the sentences you are assumed to know. Here is the main strategy for simplifying the sentence you're trying to prove:

If you're trying to prove a conditional, assume the antecedent as a premiss and try to derive the consequent. Then apply rule CP.

Here are more bottom-up strategies:

If you're trying to prove a disjunction $(\varphi \vee \psi)$, prove the conditional $(\neg\varphi \rightarrow \psi)$, then apply rule TC. (Occasionally, you can simply prove one of the disjuncts.)

If you're trying to prove a biconditional $(\varphi \leftrightarrow \psi)$, prove the two conditionals $(\varphi \rightarrow \psi)$ and $(\psi \rightarrow \varphi)$, then apply TC.

If you're trying to prove a conjunction, prove each of the conjuncts, then apply rule TC.

If you're trying to prove a universal sentence, try to prove an instance of it the sentence with a new constant. Then apply UG.

If you're trying to prove an existential sentence, try to prove an instance of it. Then apply EG.

If you're trying to prove a negated conditional $\neg(\varphi \rightarrow \psi)$, prove φ and $\neg\psi$, then apply TC.

If you're trying to prove a negated disjunction $\neg(\varphi \vee \psi)$, prove $\neg\varphi$ and $\neg\psi$, then apply TC.

If you're trying to prove a negated biconditional $\neg(\varphi \leftrightarrow \psi)$, try to prove $(\varphi \leftrightarrow \neg\psi)$, then apply TC.

If you're trying to prove negated conjunction, $\neg(\varphi \wedge \psi)$, try to prove $(\varphi \rightarrow \neg\psi)$, then apply TC.

If you're trying to prove a negated universal sentence $\neg(\forall x)\varphi$, prove $(\exists x)\neg\varphi$, then apply QE.

If you're trying to prove a negated existential sentence $\neg(\exists x)\varphi$, prove $(\forall x)\neg\varphi$, then apply QE.

If you're trying to prove a negated negation $\neg\neg\varphi$, prove φ, then apply TC.

Now we turn to the top-down strategies for simplifying the premisses and sentences you have derived from the premisses. Let me refer to the premisses and the things you have derived from the premisses as your "assumptions"; here are the strategies for simplifying assumptions:

If one of your assumptions is a conjunction, write the two conjuncts on separate lines, using TC.

If one of your assumptions is a disjunction $(\varphi \vee \psi)$ and you're trying to prove θ, prove the two conditionals $(\varphi \rightarrow \theta)$ and $(\psi \rightarrow \theta)$, then apply TC. Whenever you apply this strategy, you're sure to wind up with a pretty long proof, so use this strategy only as a last resort. Something to try first is to rewrite the disjunction as a conditional $(\neg\varphi \rightarrow \psi)$ (using TC), then to see if you can apply *modus ponens* or *modus tollens*.

If one of your assumptions is a conditional $(\varphi \rightarrow \psi)$, see if you can apply *modus ponens* or *modus tollens*. If not, rewrite the conditional as $(\neg\varphi \vee \psi)$.

If one of your assumptions is a biconditional $(\varphi \leftrightarrow \psi)$, see if you know how to prove one component, in which case you can derive the other by TC. See if you know how to prove the negation of one component, in which case you can derive the negation of the other. Otherwise, rewrite the biconditional as $((\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi))$, using TC

If one of your assumptions is an existential sentence $(\exists x)\varphi$, pick a new constant c and assume $\varphi^{x}/_{c}$ as a new premiss. Once you've proven what you're trying to prove, use ES to upgrade your premiss set, replacing $\{\varphi^{x}/_{c}\}$ by the premiss set of $(\exists x)\varphi$.

If one of your assumptions is a universal sentence $(\forall x)\varphi$, deduce $\varphi^x/_c$ for each constant c that appears in the proof.*

If one of your assumptions is a negated conjunctions $\neg(\varphi \wedge \psi\_)$, rewrite it as $(\neg\varphi \vee \neg\psi)$, using TC.

If one of your assumptions is a negated disjunction, $\neg(\varphi \vee \psi)$, rewrite it as $(\neg\varphi \wedge \neg\psi)$, using TC.

If one of you assumptions is a negated conditional $\neg(\varphi \rightarrow \psi)$, rewrite it as $(\varphi \wedge \neg\psi)$, using TC.

If one of your assumptions is a negated biconditional $\neg(\varphi \leftrightarrow \psi)$, rewrite it as $(\varphi \leftrightarrow \neg\psi)$, using TC.

If one of your assumptions is a negated existential sentence $\neg(\exists x)\varphi$, rewrite it as $(\forall x)\neg\varphi$, using QE.

If one of your assumptions is a negated universal sentence $\neg(\forall x)\varphi$, rewrite it as $(\exists x)\neg\varphi$, using QE.

If one of your assumptions is a negated negation $\neg\neg\varphi$, rewrite it as φ, using TC.

One final bottom-up rule:

If all else fails, assume the negation of what you're trying to prove and try to derive an absurdity. If you're trying to ψ, use PI to assume $\neg\psi$. If you are able to prove ψ or you are able to prove the negation of one of your other assumptions, you can use CP followed by TC to get the conclusion you want.

Let's do some examples. Let's derive "$(\exists x)(Fx \vee Hx)$" from {"$((\exists x)Fx \vee (\exists x)Gx)$," "$(\forall x)(Gx \rightarrow Hx)$"}. We follow the strategy for using a disjunctive assumption, first proving "$((\exists x)Fx \rightarrow (\exists x)(Fx \vee Hx))$" and "$((\exists x)Gx \rightarrow (\exists x)(Fx \vee Hx))$," then applying TC:

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* It generally only helps to derive $\varphi^x/_c$ for constants c that appear elsewhere in the proof. It usually does no good to instantiate with a brand new constant. The only exceptions that I know of occur when there haven't been any constants in the proof so far. An example is the derivation from the empty set of $((\forall x)Fx \rightarrow (\exists x)Fx)$:

1	1. $(\forall x)Fx$	PI
1	2. Fa	US,1
1	3. $(\exists x)Fx$	(EG),2
	4 $((\forall x)Fx \rightarrow (\exists x)Fx)$	CP,1,3

1	1. ((∃x)Fx ∨ (∃x)Gx))	PI
2	2. (∀x)(Gx → Hx)	PI
3	3. (∃x)Fx	PI
4	4. Fa	PI (for ES)
4	5. (Fa ∨ Ha)	TC,4
4	6. (∃x)(Fx ∨ Hx)	EG,5
3	7. (∃x)(Fx ∨ Hx)	ES,3,4,6
	8. ((∃x)Fx → (∃x)(Fx ∨ Hx))	CP,3,7
9	9. (∃x)Gx	PI
10	10. Gb	PI (for ES)
2	11. (Gb → Hb)	US,2
2,10	12. Hb	TC,10,11
2,10	13. (Fb ∨ Hb)	TC,12
2,10	14. (∃x)(Fx ∨ Hx)	EG,13
2.9	15. (∃x)(Fx ∨ Hx)	ES,9,10,14
2	16. ((∃x)Gx → (∃x)(Fx ∨ Hx))	CP,9,15
1,2	17. (∃x)(Fx ∨ Hx)	TC,1,8,16

As another example, let's derive "(((∃x)Ax ∨ (∃x)Bx) ↔ (∃x)(Ax ∨ Bx))" from the empty set. We prove the two directions separately. We prove the left-to-right direction by proving "((∃x)Ax → (∃x)(Ax ∨ Bx))" and "((∃x)Bx → (∃x)(Ax ∨ Bx))," then using TC; this is our general strategy for working with disjunctive assumptions. We then prove the right-to-left direction by converting the disjunction we're trying to prove to a conditional:

1	1. ((∃x)Ax ∨ (∃x)Bx)	PI (for CP)
2	2. (∃x)Ax	PI (for CP)
3	3. Aa	PI (for ES)
3	4. (Aa ∨ Ba)	TC,3
3	5. (∃x)(Ax ∨ Bx)	EG,4
2	6. (∃x)(Ax ∨ Bx)	ES,2,3,5
	7. ((∃x)Ax → (∃x)(Ax ∨ Bx))	CP,2,6
8	8. (∃x)Bx	PI (for CP)
9	9. Bb	P (for ES)
9	10. (Ab ∨ Bb)	TC,9
9	11. (∃x)(Ax ∨ Bx)	EG,10
8	12. (∃x)(Ax ∨ Bx)	ES,8,9,11
	13. ((∃x)Bx) → (∃x)(Ax ∨ Bx))	CP,8,12
1	14. (∃x)(Ax ∨ Bx)	TC,1,7,13
	15, (((∃x)Ax ∨ (∃x)Bx) → (∃x)(Ax ∨ Bx))	CP,1,14
16	16. (∃x)(Ax ∨ Bx)	PI (for CP)
17	17. (Ac ∨ Bc)	PI (for ES)
18	18. ¬(∃x)Ax	PI (for CP)
18	19. (∀x)¬Ax	QE,18
18	20. ¬Ac	US,19
17	21. (¬Ac → Bc)	TC,17
17,18	22. Bc	TC,20,21

17,18	23. $(\exists x)Bx$	EG,22
16,18	24. $(\exists x)Bx$	ES,16,17.23
16	25. $(\neg(\exists x)Ax \rightarrow (\exists x)Bx)$	CP,18,24
16	26. $((\exists x)Ax \vee (\exists x)Bx)$	TC,25
	27. $((\exists x)(Ax \vee Bx) \rightarrow ((\exists x)Ax \vee (\exists x)Bx))$	CP,16,26
	28. $(((\exists x)Ax \vee (\exists x)Bx) \leftrightarrow (\exists x)(Ax \vee Bx))$	TC,15,27

Now we derive "$((\exists x)Fx \rightarrow Ga)$" from {"$(\forall x)(Fx \rightarrow Ga)$"}:

1	1. $(\forall x)(Fx \rightarrow Ga)$	PI
2	2. $(\exists x)Fx$	PI (for CP)
3	3. Fb	PI (for ES)
1	4. $(Fb \rightarrow Ga)$	US,1
1,3	5. Ga	MP,3,4
1,2	6. Ga	ES,2,3,5
1	7. $((\exists x)Fx \rightarrow Ga)$	CP,2,6

We now do the converse proof, deriving "$(\forall x)(Fx \rightarrow Ga)$" from {"$((\exists x)Fx \rightarrow Ga)$"}:

1	1. $((\exists x)Fx \rightarrow Ga)$	PI
2	2. Fb	PI (for CP)
2	3. $(\exists x)Fx$	EG,2
1,2	4. Ga	MP 1,3
1	5. $(Fb \rightarrow Ga)$	CP,2,4
1	6. $(\forall x)(Fx \rightarrow Ga)$	UG,5

Next we formalize an argument from Lewis Carroll's *Symbolic Logic*:

> No one who really appreciates Beethoven fails to keep silence while the *Moonlight Sonata* is being played.
> Guinea pigs are hopelessly ignorant of music.
> No one who is hopelessly ignorant of music ever keeps silence while the *Moonlight Sonata* is being played.
> Therefore, guinea pigs never really appreciate Beethoven.

In symbols:

$\neg(\exists x)(Bx \wedge \neg Sx)$
$(\forall x)(Gx \rightarrow Ix)$
$\neg(\exists x)(Ix \wedge Sx)$
$\therefore \neg(\exists x)(Gx \wedge Bx)$

We derive the translated conclusion from the translated premisses, thus showing the English argument is valid:

1	1. $\neg(\exists x)(Bx \wedge \neg Sx)$	PI
2	2. $(\forall x)(Gx \rightarrow Ix)$	PI
3	3. $\neg(\exists x)(Ix \wedge Sx)$	PI
4	4. Ga	PI (for CP)
1	5. $(\forall x)\neg(Bx \wedge \neg Sx)$	QE,1
3	6. $(\forall x)\neg(Ix \wedge Sx)$	QE,3
2	7. $(Ga \rightarrow Ia)$	US,2
1	8. $\neg(Ba \wedge \neg Sa)$	US,5
3	9. $\neg(Ia \wedge Sa)$	US,6
2,4	10. Ia	MP 4,7
3	11. $(Ia \rightarrow \neg Sa)$	TC,9
2,3,4	12. $\neg Sa$	MP 10,11
1	13. $(Ba \rightarrow Sa)$	TC,8
1,2,3,4	14. $\neg Ba$	TC,12,13
1,2,3	15. $(Ga \rightarrow \neg Ba)$	CP,4,14
1,2,3	16. $\neg(Ga \wedge Ba)$	TC,15
1,2,3	17. $(\forall x)\neg(Gx \wedge Bx)$	UG,16
1,2,3	18. $\neg(\exists x)(Gx \wedge Bx)$	QE,17

Here is a more complicated example, again from Lewis Carroll:

> Animals are always mortally offended them if I fail to notice them.
> The only animals that belong to me are in that field.
> No animal can guess a conundrum unless it has been properly trained in a Board-school.
> All badgers are animals.
> None of the animals in that field are badgers.
> When a animal is mortally offended, it always rushes about wildly and howls.
> I never notice any animals unless it belongs to me.
> No animal that has been properly trained in a Board-school ever rushes about wildly and howls.
> Therefore, no badger can guess a conundrum.

In symbols:

$(\forall x)((Ax \wedge \neg Nx) \rightarrow Ox)$
$(\forall x)((Ax \wedge Mx) \rightarrow Fx)$
$\neg(\exists x)((Ax \wedge Gx) \wedge \neg Tx)$
$(\forall x)(Bx \rightarrow Ax)$
$\neg(\exists x)((Ax \wedge Fx) \wedge Bx)$
$(\forall x)((Ax \wedge Ox) \rightarrow (Rx \wedge Hx))$
$\neg(\exists x)((Ax \wedge Nx) \wedge \neg Mx)$
$\neg(\exists x)((Ax \wedge Tx) \wedge (Rx \wedge Hx))$
$\therefore \neg(\exists x)(Bx \wedge Gx)$

Now we derive the translated conclusion from the translated premisses:

1	1. (∀x)((Ax ∧ ¬Nx) → Ox)	PI
2	2. (∀x)((Ax ∧ Mx) → Fx)	PI
3	3. ¬(∃x)((Ax ∧ Gx) ∧ ¬Tx)	PI
4	4. (∀x)(Bx → Ax)	PI
5	5. ¬(∃x)((Ax ∧ Fx) ∧ Bx)	PI
6	6. (∀x)((Ax ∧ Ox) → (Rx ∧ Hx))	PI
7	7. ¬(∃x)((Ax ∧ Nx) ∧ ¬Mx)	PI
8	8. ¬(∃x)((Ax ∧ Tx) ∧ (Rx ∧ Hx))	PI
3	9. (∀x)¬((Ax ∧ Gx) ∧ ¬Tx)	QE,3
5	10. (∀x)¬((Ax ∧ Fx) ∧ Bx)	QE,5
7	11. (∀x)¬((Ax ∧ Nx) ∧ ¬Mx)	QE,7
8	12. (∀x)¬((Ax ∧ Tx) ∧ (Rx ∧ Hx))	QE,8
13	13. Ba	PI (for CP)
1	14. ((Aa ∧ ¬Na) → Oa)	US,1
2	15. ((Aa ∧ Ma) → Fa)	US,2
4	16. (Ba → Aa)	US,4
6	17. ((Aa ∧ Oa) → (Ra ∧ Ha))	US,6
3	18. ¬((Aa ∧ Ga) ∧ ¬Ta)	US,9
5	19. ¬((Aa ∧ Fa) ∧ Ba)	US,10
7	20. ¬((Aa ∧ Na) ∧ ¬Ma)	US,11
8	21. ¬((Aa ∧ Ta) ∧ (Ra ∧ Ha))	US,12
4,13	22. Aa	MP 13,16
5	23. ((Aa ∧ Fa) → ¬Ba)	TC,19
5,13	24. ¬(Aa ∧ Fa)	TC,13,23
5,13	25. (Aa → ¬Fa)	TC,24
4,5,13	26. ¬Fa	MP 22,24
2,4,5,13	27. ¬(Aa ∧ Ma)	TC,15,26
2,4,5,13	28. (Aa → ¬Ma)	TC,27
2,4,5,13	29. ¬Ma	MP 22,28
7	30. ((Aa ∧ Na) → Ma)	TC,20
2,4,5,7,13	31. ¬(Aa ∧ Na)	TC,29,30
2,4,5,7,13	32. (Aa → ¬Na)	TC,31
2,4,5,7,13	33. ¬Na	MP 22,32
2,4,5,7,13	34. (Aa ∧ ¬Na)	TC,22,32
1,2,4,5,7,13	35. Oa	TC,14,34
1,2,4,5,7,13	36. (Aa ∧ Oa)	TC,22,35
1,2,4,5,6,7,13	37. (Ra ∧ Ha)	TC,17,36
8	38. ((Aa ∧ Ta) → ¬(Ra ∧ Ha))	TC,21
1,2,4,5,6,7,8,13	39. ¬(Aa ∧ Ta)	TC,37,38
1,2,4,5,6,7,8,13	40. (Aa → ¬Ta)	TC,39
1,2,4,5,6,7,8,13	41. ¬Ta	TC,22,40
3	42. ((Aa ∧ Ga) → Ta)	TC,18
1,2,3,4,5,6,7,8,13	43. ¬(Aa ∧ Ga)	TC,41,42
1,2,3,4,5,6,7,8,13	44. (Aa → ¬Ga)	TC,43
1,2,3,4,5,6,7,8,13	45. ¬Ga	MP 22,44
1,2,3,4,5,6,7,8	46. (Ba → ¬Ga)	CP,13,45
1,2,3,4,5,6,7,8	47. ¬(Ba ∧ Ga)	TC,46
1,2,3,4,5,6,7,8	48. (∀x)¬(Bx ∧ Gx)	UG,47
1,2,3,4,5,6,7,8	49. ¬(∃x)(Bx ∧ Gx)	QE,48