MPC Completeness

There is no requirement that, within a given interpretation, every individual in the domain be named by some individual constant, and consequently there is no assurance that a universal sentence $(\forall x)\varphi$ will be true under an interpretation under which all its substitution instances (sentences got from φ by substituting individual constants for free occurrences of "x") are true. Thus we might have interpretation whose domain consists of all human beings, in which the extension of "G" is the set of Greeks and in which all the individual constants happen to denote ancient Greek philosophers. Under such an interpretation, all the substitution instances for "$(\forall x)Gx$" are true, but "$(\forall x)Gx$" isn't true.

If it just so happens that every element of $|\mathfrak{A}|$ is named by some individual constant in $\mathfrak{A}$ — in such cases, $\mathfrak{A}$ is said to be *fully named* — this it will be the case that a universal sentence is true in $\mathfrak{A}$ iff all its substitution instances are true in $\mathfrak{A}$, because we have:

$(\forall x)\varphi$ is true under $\mathfrak{A}$
iff every element of $|\mathfrak{A}|$ satisfies φ under $\mathfrak{A}$
iff (because every element of $|\mathfrak{A}|$ is named by some individual constant), for each individual constant c, the element of $|\mathfrak{A}|$ named by c in $\mathfrak{A}$ satisfies φ if $\mathfrak{A}$
iff (by the Substitution Principle) for each individual constant c, $\varphi^x/_c$ is true in $\mathfrak{A}$.

Similarly, an existential sentence is true in $\mathfrak{A}$ iff at least one of its substitution instances is true in $\mathfrak{A}$. This means that, so long as we are dealing with fully named interpretations, we can give the truth conditions for complex sentences directly in terms of the truth conditions of simpler sentences, without bringing satisfaction into the picture, thus substantially simplifying the semantic theory. If $\mathfrak{A}$ is fully named, we have:

An atomic sentence of the form Pc is true under $\mathfrak{A}$ iff $\mathfrak{A}(c) \in \mathfrak{A}(P)$.
A disjunction is true under $\mathfrak{A}$ iff one or both disjuncts are true under $\mathfrak{A}$.
A conjunction is true under $\mathfrak{A}$ iff both conjuncts are true under $\mathfrak{A}$.
A conditional is true under $\mathfrak{A}$ iff its consequent is true under $\mathfrak{A}$ or its antecedent is not.
A biconditional is true under $\mathfrak{A}$ iff both its components are true under $\mathfrak{A}$ or neither is.
A negation is true under $\mathfrak{A}$ iff its negation is not.
An existential sentence is true under $\mathfrak{A}$ iff at least one of its substitution instances is true under $\mathfrak{A}$.
A universal sentence is true under $\mathfrak{A}$ iff all its substitution instances are true under $\mathfrak{A}$.

Definition. A *complete story with witnesses* is a set of sentences Ω that meets the following seven conditions:
A disjunction is in Ω iff one or both disjuncts are.
A conjunction is in Ω iff both conjuncts are.
A conditional is in Ω iff its consequent is in Ω or its antecedent is not.

A biconditional is in Ω iff both it components are in Ω or neither is.
A negation is in Ω iff its negatum is not.
An existential sentence is in Ω iff at least on of its substitution instances is.
A universal sentence is in Ω iff all its substitution instances are.

The set of sentences true under a fully named interpretation form a complete story with instances.

Complete stories with witnesses are important because of the converse result, which was first noticed by Leon Henkin.[1] Given a complete story with witnesses Ω in a language with at least one individual constant,[2] we can construct a fully named interpretation under which all and only the members of Ω are true. The construction is not trivial, like the transition from a complete SC story to a NTA. This is because interpretations have a more complex structure than NTAs.

Given a complete story with witnesses Ω, we want to define an interpretation $\mathfrak{A}$ under which all and only the elements of Ω are true. We begin by listing the individual constants in the language in a (finite or infinite) list $\zeta_0, \zeta_1, \zeta_2, \zeta_3,\ldots$, and we stipulate that $\mathfrak{A}(\zeta_i) = i$, and that $|\mathfrak{A}|$ is to be the set of natural numbers less that n (if the language has n constants) or else the set of natural numbers (if the language has infinitely many constants).[3] Now let $\mathfrak{A}(P) = \{i\text{: } P\zeta_i \in$

[1]"The Completeness of the First-order Functional Calculus," *Journal of Symbolic Logic* 14 (1949): 159-166.

[2]If the language has no individual constants at all, then a complete story with witnesses will include every universal sentence and exclude every existential sentence. But there is no interpretation under which "$(\forall x)(Fx \wedge \neg Fx)$" is true, or under which "$(\exists x)(Fx \vee \neg Fx)$" is false. This because, when we were defining "interpretation," we chose, more-or-less arbitrarily, to exclude interpretations with the empty domain. Making the opposite choice wouldn't have made a whole lot of difference. See W. V. Quine, "Quantification and the Empty Domain," *Journal of Symbolic Logic* 19 (1954): 177-179. Reprinted in Quine, *Selected Logic Papers*, enlarged edition (Cambridge, Mass.: Harvard University Press, 1995), pp. 220-223.

[3]

I am taking for granted that, if the individual constants, if there are infinitely many of them, can be paired off with the natural numbers like this: $\zeta_0, \zeta_1, \zeta_2, \zeta_3,\ldots$. Later on, I'll assume the same thing holds true of the predicates. These two assumptions together will assure us that the sentences of the language can be arrayed in an infinite list $\xi_0, \xi_1, \xi_2, \xi_3,\ldots$. These assumptions is certainly satisfied by any language that people actually use, but there appear in pure mathematics centain purely abstract formal languages whose symbols can't be arrayed in such a list. The result that, for any complete story with witnesses, there is an interpretation under which all its members are true, and the completeness theorem in whose proof we apply the result, continue to hold in these abstract languages, but the proof has an extra twist.

$\Omega\}$, for each predicate P. Thus $P\zeta_i$ is true in $\mathfrak{A}$ iff $\mathfrak{A}(\zeta_i) \in \mathfrak{A}(P)$ iff $i \in \mathfrak{A}(P)$ iff $P\zeta_i \in \Omega$, so that an atomic sentence is true in $\mathfrak{A}$ iff it's an element of Ω. An easy induction extends this observation to all the sentences.

We want to prove the Completeness Theorem, that is, we want to prove that, if an MPC sentence χ is a logical consequence of a set Γ of MPC sentences, then there is a derivation of χ whose premiss set is included in Γ. The plan is to assume that there is no derivation of χ whose premiss set is included in Γ, and then show that there is a complete story with witnesses that includes Γ without including χ. This will show that χ isn't a logical consequence of Γ.

The proof closely resembles the corresponding proof for SC, but there are a couple of small complications arising from the fact that now we are providing witnesses. Here is one: Suppose our set Γ consists of all the sentences Gc, for c an individual constant, and let χ be $(\forall x)Gx$. Γ doesn't entail χ, and so there isn't any derivation of χ from Γ (by soundness — any sentence derivable from a set of premisses is a logical consequence of those premisses). However, there isn't any complete story with witnesses that contains all the Gcs without containing $(\forall x)Gx$. The way around this difficulty is to add infinitely many new individual constants at the beginning. Having an infinite supply of individual constants that don't appear either in Γ or in χ will give us the maneuvering room we need to construct our complete story with witnesses.

Let $\Gamma_0 = \Gamma$. By hypothesis, χ isn't derivable from Γ. We want to build up Γ_0 into a complete story with witnesses by building up larger and larger sets of sentences $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq$ ..., going through the sentences one by one, and adding the sentence if doing so doesn't disrupt the property that χ isn't derivable. The only complication is that, in order to make sure that the complete story we are forming has witnesses, whenever we add in an existential sentence, we need to add one of its substitution instances.

Enumerate all the sentences of the extended language as $\xi_0, \xi_1, \xi_2, \ldots$. Given Γ_n with the property that χ isn't derivable from Γ_n, we from the set Γ_{n+1}, as follows:

Case 1. If χ is derivable from $\Gamma_n \cup \{\xi_n\}$, let $\Gamma_{n+1} = \Gamma_n$.

Case 2. If χ isn't derivable from $\Gamma_n \cup \{\xi_n\}$ and ξ_n isn't existential, let $\Gamma_{n+1} = \Gamma_n \cup \{\xi_n\}$.

Case 3. If χ isn't derivable from $\Gamma_n \cup \{\xi_n\}$ and ξ_n has the form $(\exists x)\psi$, pick the first constant c that doesn't appear in $\Gamma_n \cup \{\xi_n, \chi\}$, and let $\Gamma_{n+1} = \Gamma_n \cup \{\xi_n, \psi^x/_c\}$. We can do this because at the very beginning we introduced an infinite supply of new constants.

We want to persuade ourselves that χ isn't derivable from Γ_{n+1}. Cases 1 and 2 are trivial. For Case 3, notice that, if we had a derivation of χ whose premiss set included $\psi^x/_c$, together with various members of $\Gamma_n \cup \{\xi_n\}$, then we could utilize rule ES to replace $\psi^x/_c$ in our premiss set by $(\exists x)\psi$. But this would give us a derivation of χ whose premiss set was contained in $\Gamma_n \cup \{\xi_n\}$, contrary to hypothesis.

Let Γ_∞ be the union of the Γ_ns. Then χ isn't derivable from Γ_∞, but whenever a sentence φ isn't in Γ_∞, χ is derivable from $\Gamma_\infty \cup \{\varphi\}$. It follows that every sentence derivable from Γ_∞ is an element of Γ_∞, and also that, whenever φ isn't an element of Γ_∞, $(\varphi \rightarrow \chi)$ is an element of Γ_∞. Also, whenever Γ_∞ contains an existential sentence, it contains a substitution instance of the existential sentence; if the existential sentence is ξ_k, then a substitution instance of the existential sentence is introduced in Γ_{k+1}.

We need to see that Γ_∞ is a complete story with witnesses. The proofs of the first five clauses in the definitions of "complete story with witnesses" — that is, the proof that Γ_∞ is a complete story — are unchanged from the sentential calculus, and there's no point in repeating them here. We just noted that if an existential sentence is in Γ_∞, then at least one substitution instance is in Γ_∞. That an existential sentence is in Γ_∞ if any of its instances are follows from the fact that the existential sentence is derivable from each of its instances by EG. This leaves the clause for universal sentences. In proving it, we take advantage of the fact that we already have the first six clauses in the definition of "complete story with witnesses":

$(\forall x)\varphi$ is in Γ_∞
iff $\neg(\forall x)\varphi$ isn't in Γ_∞ (by the fifth clause)
iff $(\exists x)\neg\varphi$ isn't in Γ_∞ (because Γ_∞ is closed under rule ES)
iff no substitution instance of $(\exists x)\neg\varphi$ is in Γ_∞ (by the sixth clause)
iff no sentence of the form $\neg\varphi^{x}/_{c}$ is in Γ_∞
iff every sentence of the form $\varphi^{x}/_{c}$ is in Γ_∞
iff every substitution instance of $(\forall x)\varphi$ is in Γ_∞.

Compactness Theorem. A set of MPC sentences is consistent if every finite subset is consistent.

The proof is unchanged from the sentential calculus.

Löwenheim-Skolem Theorem. Every consistent set of MPC sentences has model whose domain is the set of natural numbers.

If Γ is consistent, then we can produce, in a language obtained from the language of Γ by adding infinitely many new individual constants, a complete story with witnesses that includes Γ. The Henkin model for this complete story with witnesses has the set of natural numbers as its domain.[4]

[4]Unlike our other results, the Löwenheim-Skolem theorem does depend on the assumption that the sentences can be put into one-one correspondence with the natural numbers.