Logic I - Session 24

Completeness of PD

Last time

- We started to prove that PD is complete.
- We were working on proving that an ES-variant of any C-PD set is a subset of an MC-∃C-PD set Γ^*.
- We set out a procedure for constructing Γ^*, and proved that Γ^* was consistent.
- But there's one thing I need to correct...

Last time

- Alex asked last time why we need to deal with evenly subscripted sets. In my answer, I misidentified the reason.
- Our procedure for building Γ* included this clause:
  - If Γi ∪ {**Pi**} is C-PD, and **Pi** is of the form (∃**x**)**Q**:

    Γi+1 is Γi ∪ {**Pi**, **Q(a/x)**}, where **a** is the alphabetically

    earliest constant not occurring in **Pi** or any member of Γi

- I asked, how do we know we can add **Q(a/x)** to Γk ∪ {**Pi**}?
- The answer someone gave, and I agreed with, relied on the thought that finitely constants appear in any Γi and **Pi**.
  - We were wrong.

Last time

- Our main goal is to show that if $\Gamma \vDash \mathbf{P}$ then $\Gamma \vdash \mathbf{P}$.
- We do *not* assume initially that Γ is finite --- that's why compactness is an interesting result.
- So we do *not* assume that $\Gamma \cup \{\sim\mathbf{P}\}$ is finite either!
- So Γ might contain infinitely many constants.

Last time

- Now, in our construction of Γ^*, we sometimes need to add **Q(a/x)** to Γk, where **a** is a constant that isn't in any sentences in the set.
  - How do we know this is possible?
  - The answer explains why we bother transforming $\Gamma \cup \{\sim$**P**$\}$ into an evenly subscripted set.

- Let's now turn back to proving that our construction of Γ^* yields a set that's MC-∃C-PD.
- We proved that Γ^* is consistent. Let's now prove that it's maximally consistent.

An ES-variant of $\Gamma \cup \{\sim\mathbf{P}\}$ is C-PD

↓

Any ES-variant of $\Gamma \cup \{\sim\mathbf{P}\} \subseteq$ a MC-∃C-PD set Γ^* (11.4.4)

- Goal: Show that Γ^* is maximally consistent.
- Suppose the contrary: There's a $\mathbf{P}k \notin \Gamma^*$ s.t. $\Gamma^* \cup \{\mathbf{P}k\}$ is C-PD.
  - Since $\mathbf{P}k$ is a PL sentence, it occurs kth in our enumeration.
  - By the def. of our Γ-sequence, $\Gamma k+1 = \Gamma k \cup \{\mathbf{P}k\}$ if that's C-PD.
  - $\Gamma k \cup \{\mathbf{P}k\}$ **is** C-PD.
    - Since if $\{\mathbf{P}k\}$ were inconsistent with Γk, it would be inconsistent with every superset of Γk, e.g. Γ^*.
  - So $\Gamma k+1 = \Gamma k \cup \{\mathbf{P}k\}$ (...perhaps plus a substitution instance)
  - But that means $\mathbf{P}k \in \Gamma k+1$, so because $\Gamma k+1 \subseteq \Gamma^*$, $\mathbf{P}k \in \Gamma^*$.
  - Contradiction.
- So there's no $\mathbf{P}k \notin \Gamma^*$ s.t. $\Gamma^* \cup \{\mathbf{P}k\}$ is C-PD. I.e. Γ^* is maximal.

An ES-variant of $\Gamma \cup \{\sim \mathbf{P}\}$ is C-PD

↓

Any ES-variant of $\Gamma \cup \{\sim \mathbf{P}\} \subseteq$ a MC-∃C-PD set Γ^* (11.4.4)

- Finally, let's show that Γ^* is existentially complete.
  - I.e., that for each sentence in Γ^* of the form **(∃x)Q**, there's a substitution instance of **(∃x)Q** in Γ^*.
- So suppose **(∃x)Q** is in Γ^*.
  - **(∃x)Q** is in our enumeration, at some position k.
  - $\Gamma k \cup \{\mathbf{P}k\}$, i.e. $\Gamma k \cup \{(\exists \mathbf{x})\mathbf{Q}\}$, is either C-PD or IC-PD.
  - If it were IC-PD, then since **(∃x)Q** is in Γ^*, Γ^* would be IC-PD.
  - So $\Gamma k \cup \{(\exists \mathbf{x})\mathbf{Q}\}$ is C-PD.
  - But that means $\Gamma k+1$ is $\Gamma k \cup \{(\exists \mathbf{x})\mathbf{Q}, \mathbf{Q}(\mathbf{a}/\mathbf{x})\}$, for some **a**.
- So if **(∃x)Q** is in Γ^*, so is some substitution instance.

Completeness

$\Gamma \nvdash$ **P**

↓ ☑

An ES-variant of $\Gamma \cup \{\sim$**P**$\}$ is C-PD

↓ ☑

Any ES-variant of $\Gamma \cup \{\sim$**P**$\} \subseteq$ a MC-∃C-PD set Γ^* (11.4.4)

↓

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

↓ ☑

$\Gamma \cup \{\sim$**P**$\} \subseteq$ a Q-C set Γ^*

↓ ☑

$\Gamma \cup \{\sim$**P**$\}$ is Q-C

↓ ☑

$\Gamma \nvDash$ **P**

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- Suppose Γ* is MC-PD and ∃C.
- We'll first prove some things about the membership of Γ*, then we'll give an interpretation on which every member is true.
- This will show that Γ* is Q-C.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- First, a helpful result: $\Gamma^* \vdash \mathbf{Q}$ iff $\mathbf{Q} \in \Gamma^*$.
  - Suppose $\Gamma^* \vdash \mathbf{Q}$.
  - Suppose $\mathbf{Q} \notin \Gamma^*$. Then since Γ^* is MC, $\Gamma^* \cup \{\mathbf{Q}\}$ is IC.
  - Then $\Gamma^* \cup \{\mathbf{Q}\} \vdash \mathbf{R} \& {\sim}\mathbf{R}$.
  - So by ~I, $\Gamma^* \vdash {\sim}\mathbf{Q}$.
  - So $\Gamma^* \vdash \mathbf{Q} \& {\sim}\mathbf{Q}$, contradicting the fact that Γ^* is MC.
- So if $\Gamma^* \vdash \mathbf{Q}$ then $\mathbf{Q} \in \Gamma^*$.
  - Now suppose $\mathbf{Q} \in \Gamma^*$.
  - Then since $\{\mathbf{Q}\} \vdash \mathbf{Q}$, and $\{\mathbf{Q}\} \subseteq \Gamma^*$, $\Gamma^* \vdash \mathbf{Q}$.
- So if $\mathbf{Q} \in \Gamma^*$, then $\Gamma^* \vdash \mathbf{Q}$.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- Now let's begin proving facts about the membership of Γ^*.
    - Suppose **P**$\in\Gamma^*$.
        - Suppose ~**P**$\in\Gamma^*$.
        - Then Γ^* is IC-PD, contradicting what we've proved.
    - So ~**P**$\notin\Gamma^*$.
- So if **P**$\in\Gamma^*$ then ~**P**$\notin\Gamma^*$. And contraposing: if ~**P**$\in\Gamma^*$ then **P**$\notin\Gamma^*$.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- Suppose **P**$\notin\Gamma^*$.
  - Since Γ^* is MC, $\Gamma^*\cup\{$**P**$\}$ is IC-PD.
  - So $\Gamma^*\cup\{$**P**$\} \vdash$ **Q**&~**Q**.
  - But then $\Gamma^*\cup\{$~~**P**$\} \vdash$ **Q**&~**Q**. So $\Gamma^*\cup\{$~~**P**$\}$ is IC-PD.
  - So by ~E, $\Gamma^* \vdash$ ~**P**.
  - We proved that if $\Gamma^*\vdash$**Q**, then **Q**$\in\Gamma^*$.
  - So since $\Gamma^* \vdash$ ~**P**, ~**P**$\in\Gamma^*$.
- So if **P**$\notin\Gamma^*$, then ~**P**$\in\Gamma^*$. And contraposing: if ~**P**$\notin\Gamma^*$ then **P**$\in\Gamma^*$.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- So, we have: **P**$\in\Gamma^*$ iff ~**P**$\notin\Gamma^*$ and, equivalently, **P**$\notin\Gamma^*$ iff ~**P**$\in\Gamma^*$.
- Similar results parallel our proofs about SD, e.g.:
  - **P**&**Q**$\in\Gamma^*$ iff **P**$\in\Gamma^*$ and **Q**$\in\Gamma^*$.
  - **P**v**Q**$\in\Gamma^*$ iff **P**$\in\Gamma^*$ or **Q**$\in\Gamma^*$.
  - **P**⊃**Q**$\in\Gamma^*$ iff **P**$\notin\Gamma^*$ or **Q**$\in\Gamma^*$.
  - **P**≡**Q**$\in\Gamma^*$ iff **P**$\in\Gamma^*$ and **Q**$\in\Gamma^*$, or **P**$\notin\Gamma^*$ and **Q**$\notin\Gamma^*$.
- New results will concern quantified sentences and their substitution instances.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- To prove: **(∃x)P**∈Γ* iff for some constant **a**, **P(a/x)**∈Γ*.
  - Suppose **(∃x)P**∈Γ*.
  - Since Γ* is ∃C, there's some **a** such that **P(a/x)**∈Γ*.
- So if **(∃x)P**∈Γ*, then there's some **a** such that **P(a/x)**∈Γ*.
  - Now suppose for some constant **a**, **P(a/x)**∈Γ*.
  - Since {**P(a/x)**} ⊢ **(∃x)P** and **P(a/x)**∈Γ*, we have Γ* ⊢ **(∃x)P.**
  - We proved: Γ*⊢**Q** iff **Q**∈Γ*.
  - So **(∃x)P**∈Γ*.
- So if there's some **a** such that **P(a/x)**∈Γ*, then **(∃x)P**∈Γ*.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- So we've shown: **(∃x)P**∈Γ* iff for some constant **a**, **P(a/x)**∈Γ*.
- We can prove a similar result for universally quantified sentences.

- To prove: **(∀x)P**∈Γ* iff for every constant **a**, **P(a/x)**∈Γ*.
  - Suppose **(∀x)P**∈Γ*.
  - For any **a**, since {**(∀x)P**}⊢**P(a/x)**, Γ*⊢**P(a/x)**.
  - We proved earlier that if Γ*⊢**Q**, then **Q**∈Γ*.
  - So **P(a/x)**∈Γ*.
- So if **(∀x)P**∈Γ*, then for every constant **a**, **P(a/x)**∈Γ*.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- Now suppose: For every constant **a**, **P(a/x)**$\in\Gamma^*$.
  - Suppose **(∀x)P**$\notin\Gamma^*$.
  - We proved that <u>if **P**$\notin\Gamma^*$, then ~**P**$\in\Gamma^*$.</u>
  - So ~**(∀x)P**$\in\Gamma^*$.
  - {~**(∀x)P**} ⊢ **(∃x)**~**P**, so $\Gamma^*\vdash$ **(∃x)**~**P**.
  - We proved earlier that if $\Gamma^*\vdash$**Q**, then **Q**$\in\Gamma^*$.
  - So **(∃x)**~**P** $\in$ Γ^*.
  - Since Γ^* is ∃C, now there's some **a** such that ~**P(a/x)**$\in\Gamma^*$.
  - Then **P(a/x)**$\notin\Gamma^*$! Contradicts our assumption.
- So it's not true that **(∀x)P**$\notin\Gamma^*$. I.e., **(∀x)P**$\in\Gamma^*$.

- So if, for every constant **a**, **P(a/x)**$\in\Gamma^*$, then **(∀x)P**$\in\Gamma^*$.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- So here are our results about the membership of Γ^*.
  - $\mathbf{P} \in \Gamma^*$ iff $\sim\mathbf{P} \notin \Gamma^*$
  - $\mathbf{P} \& \mathbf{Q} \in \Gamma^*$ iff $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$.
  - $\mathbf{P} \vee \mathbf{Q} \in \Gamma^*$ iff $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
  - $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$ iff $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
  - $\mathbf{P} \equiv \mathbf{Q} \in \Gamma^*$ iff $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$, or $\mathbf{P} \notin \Gamma^*$ and $\mathbf{Q} \notin \Gamma^*$.
  - $(\exists \mathbf{x})\mathbf{P} \in \Gamma^*$ iff for some constant $\mathbf{a}$, $\mathbf{P(a/x)} \in \Gamma^*$.
  - $(\forall \mathbf{x})\mathbf{P} \in \Gamma^*$ iff for every constant $\mathbf{a}$, $\mathbf{P(a/x)} \in \Gamma^*$.
- This will allow us to prove that Γ^* is Q-C by specifying an interpretation that makes all its members true.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- Now recall, in proving the analogous result for SD, we just specified a truth-value assignment.
- But interpretations for PL are more complicated, so we'll need to do more.
- In order to give an interpretation that works, let's first remember that can alphabetically enumerate the individual constants of PL.

- For any n, we have constants at positions n, n+1, … in our enumeration.
- Each constant can be thus be associated with a different unique positive integer.
  - a with 1, b with 2, … v with 22, a1 with 23, … v4 with 88, …

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- I* will be the following interpretation:
  - UD = {n | n is a positive integer}
  - For each sentence letter **P**, I*(**P**) = T iff **P**∈Γ*. (Just like SD!)
  - For each i.c. **a**, I*(**a**) is the integer associated with **a**.
    - a: 1
    - v4: 88

    ...
  - For each **n**-place predicate letter **A**, I*(**A**) includes all and only the **n**-tuples <I*($\mathbf{a}_1$), ..., I*($\mathbf{a}_n$)> such that $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n \in \Gamma^*$.
- We'll show that every member of Γ* is true on I*, by proving that for any sentence **P** of PL, **P** is true on I* iff **P**∈Γ*.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- Our proof will be by mathematical induction on the number of OLOs in sentences.
- **Basis Clause**: For each sentence **P** with zero OLOs,

  P is true on I* iff **P** $\in \Gamma^*$.

  - Proof: **P** is either a sentence letter or atomic formula.
  - If **P** is a sentence letter, I*(**P**)=T iff **P**$\in\Gamma^*$, by def. of I*.
  - If **P** is an atomic formula, then **P** is of the form $\mathbf{Aa_1 \ldots a_n}$.
    - <u><I*($\mathbf{a_1}$), ..., I*($\mathbf{a_n}$)> $\in$ I*(**A**)</u> iff $\mathbf{Aa_1 \ldots a_n} \in \Gamma^*$. [by def. of I*]
    - $\mathbf{Aa_1 \ldots a_n}$ is true on I* iff <u><I*($\mathbf{a_1}$), ..., I*($\mathbf{a_n}$)> $\in$ I*(**A**)</u>.
    - So $\mathbf{Aa_1 \ldots a_n}$ is true on I* iff $\mathbf{Aa_1 \ldots a_n} \in \Gamma^*$.
  - So if **P** has zero OLOs, **P** is true on I* iff **P**$\in\Gamma^*$.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- **Inductive step**: If for each sentence **P** with k OLOs, **P** is true on I* iff **P** ∈ Γ*, then for each sentence **P** with k+1 OLOs, **P** is true on I* iff **P** ∈ Γ*.
- Proof: Suppose that for each sentence **P** with k OLOs, **P** is true on I* iff **P** ∈ Γ*.
- Then **P** has one of the following seven forms: ~**Q**, **Q&R**, **QvR**, **Q⊃R**, **Q≡R**, **(∀x)Q**, or **(∃x)Q**.
- Thus there are 7 cases to test.
- The cases where **P** is a TF-compound are similar to cases we covered in discussing SD, so we'll only cover one.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- Suppose **P** is of the form **QvR**.
- To prove: **P** is true on I* iff **P**∈Γ*.
  - Suppose **P** is true on I*. Then either **Q** or **R** is true on I*.
  - **Q** and **R** each have k or fewer OLOs.
  - So by the inductive hypothesis, **Q** is true on I* iff **Q**∈Γ*, and **R** is true on I* iff **R**∈Γ*.
  - In each case, either **Q**∈Γ* or **R**∈Γ*.
  - Earlier, we noted that **Q**v**R**∈Γ* iff **Q**∈Γ* or **R**∈Γ*.
  - So **Q**v**R**∈Γ*. I.e., **P**∈Γ*.
- So if **P** is true on I*, then **P**∈Γ*.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- Now suppose **P**$\in\Gamma^*$. I.e. **QvR**$\in\Gamma^*$.
- Now **Q**$\in\Gamma^*$ or **R**$\in\Gamma^*$, since **Q**v**R**$\in\Gamma^*$ iff **Q**$\in\Gamma^*$ or **R**$\in\Gamma^*$.
- By the inductive hypothesis, **Q** is true on I* iff **Q**$\in\Gamma^*$, and **R** is true on I* iff **R**$\in\Gamma^*$.
- So either **Q** or **R** is true on I*.
- If either **Q** or **R** is true on I*, **QvR** is true on I*.
- So **QvR** is true on I*. I.e. **P** is true on I*.

- So if **P** is of the form QvR, then **P** is true on I* iff **P**$\in\Gamma^*$.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- We've supposed:
  - For each sentence **P** with k OLOs, **P** is true on I* iff **P** $\in \Gamma^*$.
- And we're trying to prove:
  - For each sentence **P** with k+1 OLOs, **P** is true on I* iff **P** $\in \Gamma^*$.
- We're working by cases, for each form **P** could have.
- Next we'll turn to cases where **P** is quantified.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- Suppose **P** is of the form **(∀x)Q**.
- To prove: **P** is true on I* iff **P**∈Γ*.
  - Suppose **P** is true on I*. I.e., **(∀x)Q** is true on I*.
  - Then for every **a**, **Q(a/x)** is true on I*.
  - For every **a**, **Q(a/x)** has k or fewer OLOs, and hence by the inductive hypothesis, for every **a**, **Q(a/x)**∈Γ*.
  - We earlier showed that **(∀x)P**∈Γ* iff for every **a**, **P(a/x)**∈Γ*.
  - So **(∀x)Q**∈Γ*. I.e. **P**∈Γ*.
- So if **P** is true on I*, then **P**∈Γ*.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- Suppose **P** is false on I*. I.e. **(∀x)Q** is false on I*.
- **(∀x)Q** is false on I* iff for some ic **a**, **Q(a/x)** is false on I*. (?)
  - (Every object in the UD of I* is named by some constant.)
- So for some ic **a**, **Q(a/x)** is false on I*.
- For every ic **a**, **Q(a/x)** has k or fewer OLOs, and hence by the inductive hyp., **Q(a/x)** is false on I* iff **Q(a/x)**∉Γ*.
- So for some ic **a**, **Q(a/x)**∉Γ*.
- Again, we know that **(∀x)Q**∈Γ* iff for every ic **a**, **Q(a/x)**∈Γ*.
- So **(∀x)Q**∉Γ*. I.e. **P**∉Γ*.

- So we've shown that if **P** is of the form **(∀x)Q**, **P** is true on I* iff **P**∈Γ*.

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

- I won't go through the case for **P** of the form **(∃x)Q**, but the proof is similar. The completed mathematical induction would be:

- **Basis Clause**: For each SL sentence **P** with zero OLOs,
  P is true on I* iff **P** ∈ Γ*.

- **Inductive step**: If for each SL sentence **P** with k OLOs, **P** is true on I* iff **P** ∈ Γ*, then for each SL sentence **P** with k+1 OLOs, **P** is true on I* iff **P** ∈ Γ*.

- **Conclusion**: For any SL sentence **P**, **P** is true on I* iff **P**∈Γ*.

- So there's an interpretation that mem Γ* true. So Γ* is Q-C.

Completeness

$\Gamma \nvdash$ **P**

↓ ☑

An ES-variant of $\Gamma \cup \{\sim$**P**$\}$ is C-PD

↓ ☑

Any ES-variant of $\Gamma \cup \{\sim$**P**$\} \subseteq$ a MC-∃C-PD set Γ^* (11.4.4)

↓ ☑

If Γ^* is MC-∃C-PD then Γ^* is Q-C (11.4.8)

↓ ☑

$\Gamma \cup \{\sim$**P**$\} \subseteq$ a Q-C set Γ^*

↓ ☑

$\Gamma \cup \{\sim$**P**$\}$ is Q-C

↓ ☑

$\Gamma \nvDash$ **P**

24.241 Logic I
Fall 2009