Philosophy 244: #7—Modal Metalogic: Completeness

Soundness for a system S says that its theorems are S-valid, valid in all S-frames. Completeness says that every S-valid wff is provable in S. The two together show system S is *adequate*. You might think this is no great accomplishment; why would it be called S-validity if it wasn't going to line up with provability in S? For provability in S to agree with S-validity is about as informative as being told that Lou Gehrig died of Lou Gehrig's disease. What we learn is that there's a distinctive form of validity corresponding to system T. Given how validity was defined we learn that there's a distinctive sort of frame. The surprise isn't that the relevant frames are called T-frames; it's that there's that kind of frame in the first place.

Well, but that is somewhat informative; we learn that he had a disease and it was that disease that killed him; he didn't die of boredom or whatever. Adequacy results are informative in a similar way.

Systems can be defined either in terms of an axiomatic basis, or in terms of theorems; the first approach is more fine-grained since there can be two bases for the same set of theorems. For the most part we'll be using the second notion. A system is a set of wffs. Not any old set, e.g., not the wffs of length 51! The set has to be closed under some kind of implication. Our focus will be on *extensions of K*, that is, the theorems provable by K-rules from K and A = a set of axioms. These are called *normal* systems.

Not all modal systems have associated frames. KH = K + $\Box(\Box p \equiv p) \supset p$ doesn't (160-1)

Frames and Validity

If $\mathcal{C}$ is a class of frames, however heterogeneous, there's an associated notion of validity: α is $\mathcal{C}$-valid iff it's valid on every frame in the class, that is, true in every world of every model based on a frame. in $\mathcal{C}$. Sometimes we'll speak too of validity in a model, understood as truth in every world of the model.

> Main Idea: For every consistent normal system S, there is a particular model $\mathcal{M}_S$, "the canonical model of S," such that α is a theorem of S iff α is valid in $\mathcal{M}_S$.

The reason this matters for completeness is this. We want to show that all $\mathcal{C}$-valid formulae are theorems of system S. Suppose we can show that $\mathcal{C}$ contains the frame of this canonical model $\mathcal{M}_S$. Then the $\mathcal{C}$-validity of α entails that α is valid in the canonical model, and so by our lemma that α is a theorem of S.

So, how to construct the canonical model? Worlds can be anything we like; the domain of worlds is completely unconstrained as we're doing it. No harm then if we make them sets of wffs, the very wffs that are going to come out true in that world on the canonical model. Not any old set of wffs is fit to play the role of truths in some world. The set better be consistent, certainly. Second it better be "self-satisfying,"—if it contains a disjunction it contains one of the disjuncts, and vice versa. Third it better be "opinionated," in the sense of not remaining agnostic about α's truth-value for any wff α.

This all boils down to the set being "maximal consistent." A set Λ of wffs is S-*inconsistent* iff it contains α_i such that $\vdash_S \neg(\alpha_1\&...\&\alpha_n)$. Λ is *consistent* in S iff this never happens, ie., $\neg(\alpha_1\&...\&\alpha_n)$ is never a theorem of S for any α_is in Λ you care to pick.

A set Λ of wffs is *maximal* iff for every wff α, it contains either α or its negation. either it or its negation belongs to the set. A set of wffs is *maximal consistent* (w.r.t. S) iff it has both of these features.

Lemma 6.1 Spose that Λ is maximal consistent w.r.t. S. Then

6.1a exactly one of α, $\neg\alpha$ is in Λ

6.1b $\alpha\vee\beta$ is in Λ iff at least one of α,β is in Λ

6.1c $\alpha\wedge\beta$ is in Λ iff both of α,β are in Λ

6.1d if α, $\alpha\supset\beta$ are in Λ then so is β

Proof: (a) One is in by maximality. If both were in then Λ would be inconsistent. Etc.

Lemma 6.2 Spose Λ is maximal consistent w.r.t. S. Then
6.2a if $\vdash_S \alpha$ then $\alpha\epsilon\Lambda$
6.2b if $\vdash_S \alpha\supset\beta$ then ($\alpha\epsilon\Lambda$ only if $\beta\epsilon\Lambda$).

Proof: (a) If $\vdash_S \alpha$, then $\neg\alpha$ is S-inconsistent, so $\alpha\epsilon\Lambda$ by 6.1a (b) $\alpha\supset\beta\ \epsilon\ \Lambda$ by 6.2a. By 6.1d, if $\alpha\epsilon\Lambda$ so is β. (b).....?

Prop. 6.3 If Γ is S-consistent, then it has a maximal consistent extension, that is, there's a maximal consistent Λ s.t. $\Gamma \subseteq \Lambda$.

Proof: List the wffs (all infinitely many of them) of modal propositional logic as α_1, α_2,...Define a sequence Γ_0, Γ_1, etc. of sets of wffs as follows:
(1) $\Gamma_0 = \Gamma$
(2) $\Gamma_{n+1} = \Gamma_n \cup \{\alpha_{n+1}\}$ if that's S-consistent, otherwise $\Gamma_{n+1} = \Gamma_n \cup \{\neg\alpha_{n+1}\}$.
Clearly if Γ_n is S-consistent, so is Γ_{n+1}. Otherwise Γ_n would be inconsistent both with α and $\neg\alpha$; so it would entail their negations, which makes it itself inconsistent. $\Lambda =$ the union of the Γ_ns is consistent; if not it would have an inconsistent finite subset, whence one of the Γ_ns would be inconsistent. Γ is maximal since for any wff α_n, either it or its negation was added at the *n*th stage.

Accessibility

Now we set up the accessibility relation. For any set Γ of wffs, let $\Box^-(\Lambda) = \{\beta \mid \Box\beta\epsilon\ \Gamma\}$. We'll say that Γ bears R to Δ iff $\Box^-(\Gamma)\subseteq \Delta$. (If $\Box\beta$ holds at Γ, then we want Γ to "see" Δ iff β holds at Δ.) The next Lemma is for the case where Γ contains instead $\neg\Box\beta$; we need some accessible Δ to contain $\neg\beta$. The result will be that if Γ calls β necessary, all the worlds it can see are β-worlds, whereas if it calls β unnecessary, it can see a $\neg\beta$-world.

Lemma 6.4 Given a normal system S of propositional modal logic, let Γ be an S-consistent set of wffs containing $\neg\Box\alpha$. Then $\Box^-(\Gamma) \cup \{\neg\alpha\}$ is S-consistent.

Proof. If not, there are β_i in $\Box^-(\Gamma)$ such that $\vdash_S \neg(\beta_1\wedge \ldots.\beta_n\wedge\neg\alpha)$, which means that $\vdash_S(\beta_1\wedge\ldots.\beta_n)\supset\alpha$. By DR1, $\vdash_S \Box(\beta_1\wedge\ldots.\beta_n)\supset\Box\alpha$. $\Box$-distribution gives $\vdash_S(\Box\beta_1\wedge \ldots\Box\beta_n)\supset\Box\alpha$. But then $\Box\beta_1$,..,$\Box\beta_n$, $\neg\Box\alpha$ is not S-consistent. It is a subset of Γ though (why?), so Γ is not S-consistent either, which was to be shown.

Canonical models

The canonical model for S is <W,R,V>, where

W = $\{\Gamma \mid \Gamma$ is maximal consistent w.r.t. S$\}$
*w*R*u* iff $\beta\ \epsilon$ *u* whenever $\Box\beta\ \epsilon$ *w*, that is, $\Box^-$(*w*) $\subseteq$ *u*.
V(*p*,*w*)=1 iff *p*ϵ*w*

Prop. 6.5 If <WRV> = the canonical model for normal system S, then V(α,*w*)=1 iff $\alpha\ \epsilon$ *w*.

Proof By induction on the complexity of α.

Cor.6.6 α is valid in the canonical model for S iff α is a theorem of S.

Proof: (i) If $\vdash_S \alpha$ then α is in every maximal S-consistent set, by 6.2a, Hence α is in each world in the canonical model's W. 6.5 then tells is that V(α,*w*)=1 for each *w*ϵ*W*, ie., α is valid in the canonical model. (ii) Spose that α is not a theorem of S. Then $\neg\alpha$ is S-consistent so by 6.3 there's a maximal S-consistent set *w* that contains it. V(α,*w*)=0 by 6.5. So α is not valid in the canonical model.

Where does this leave us? To establish completeness it's enough now to focus on the canonical model. This makes the job much easier.

To establish that	It's enough to show that
K is complete	the canonical model for K is built on some frame or other
D is complete	the canonical model for D is built on a serial frame
T is complete	the canonical model for T is built on a reflexive frame
B is complete	the canonical model for B is built on a reflexive, symmetric frame
S4 is complete	the canonical model for S4 is built on a reflexive, transitive frame
S5 is complete	the canonical model for S5 is built on a reflexive, transitive, symmetric frame

The argument goes like this, illustrating with T.

1. Suppose α is valid over the class $\mathcal{C}$ of reflexive frames (it's T-valid)
2. The canonical model for T is built on a reflexive frame, we're going to show!
3. So α is valid in the canonical model for system T. (1,2)
4. Corollary 6.6 says validity in the canonical model for T suffices for provability in T.
5. So α is provable in T. (3,4)

The one piece that has to be filled in is that the canonical model for T is built on a reflexive frame, that is, the accessibility relation R in that model is reflexive. How did we define it? We said that Γ bears R to Δ iff $\Box^{-}(\Gamma)\subseteq\Delta$, Γ and Δ being maximal consistent in T. $\Box^{-}(\Gamma)\subseteq\Delta$ iff whenever $\Box\alpha \,\epsilon\, \Gamma$, α lies within Δ. Right now we're interested in the case where $\Delta = \Gamma$.

The question then becomes this: when $\Box\alpha \,\epsilon\, \Gamma$, must α lie within Γ too? All we have to go on is that Γ is a maximal T-consistent set. But that's enough. Γ contains $\Box\alpha \supset \alpha$ because a theorem of T is going to belong to *every* maximal T-consistent set, by 6.2a. Γ thus contains $\Box\alpha$ and $\Box\alpha \supset \alpha$. Ah, but it is closed under modus ponens by 6.2b. So it contains α too, as promised.

All the systems go this way more or less. How do we know that the canonical model for S4 is built on a reeflexive, transitive frame? We have to show that if Γ, Δ, and Ξ, are maximal S4-consistent sets, then, first, α is in Γ if $\Box\alpha$ is (that's been done), and second, for transitivity,

If Δ contains α whenever Γ contains $\Box\alpha$, and

Ξ contains α whenever Δ contains $\Box\alpha$, then

Ξ contains α whenever Γ contains $\Box\alpha$.

How would we show this?

Step One: What else must Γ contain if it contains $\Box\alpha$?
Step Two: ....
Step Three: ....