Philosophy 244: #8— Counterfactuals, Neighborhood Semantics, Probability, Predicative Necessity, etc.

Modal operators are non-truth-functional; the truth-value of $\Box\alpha$ at a world is not determined by α's s truth-value at that world. Are modal operators X-functional for any other X? Yes. If you let the truth-set $|\alpha|$ of α in a given model be the set of worlds (in that model) at which α true, then the truth-value, and indeed truth-set of $\Box\alpha$ is determined by that of α . Let R(*w*) be the set of worlds *w* bears R to; then

$\Box\alpha$ is true at *w* iff R(*w*) $\subseteq |\alpha|$.

$\Diamond\alpha$ is true at *w* iff R(*w*) overlaps $|A|$— R(*w*) $\cap\ |\alpha| \neq \varnothing$

Neighborhood semantics

Once you see that the clauses can be written this way it opens your mind a bit. Why not an operator Δ such that

$\Delta\alpha$ is true at *w* iff R(α) $= |\alpha|$

$\Delta\alpha$ is true at *w* ff $|\alpha| \subseteq$ R(*w*)

$\Delta\alpha$ is true at *w* iff R(*w*) $\cap\ |\alpha|$ contains exactly one world

To do this in full generality, we replace R, a relation between worlds, with $\mathcal{R}$, a relation worlds bear to sets of worlds. Rules for truth take the form

$\Delta\alpha$ is true at *w* iff $w\mathcal{R}|\alpha|$

This style of modal semantics is called *neighborhood* semantics. <W,$\mathcal{R}$> is a neighborhood frame. Regular old relational frames can be treated as the special case of neighborhood frames where *w* bearing $\mathcal{R}$ to a set corresponds to its bearing R to every member of the set. Neighborhood semantics extends to binary modal operators as follows:

α * β is true at *w* iff $\mathcal{R}$(*w*, $|\alpha|$,$|\beta|$)

One such binary operator is the *strict* conditional $\Rightarrow$. The intended meaning of $\alpha \Rightarrow \beta$ is that α implies β, or *Necessarily if* α *then* β. Given that *Necessarily, if p&q, then p*, we'd expect it to be true at each world that (*p*&*q*)$\Rightarrow$ *p*. The truth-rule that gets us this result is

$\alpha \Rightarrow \beta$ is true at *w* iff $|\alpha| \subseteq |\beta|$

Or maybe $|\alpha|\cap$X $\subseteq |\beta|\cap$X for some suitable X. Let's ignore this complication.

Counterfactual Conditionals

Another is the *counterfactual* conditional $\Box\!\!\rightarrow$. Read $\alpha\Box\!\!\rightarrow\beta$ as *if it were to be that* α*, then it would be that* β*. If you'd won the lottery, you'd be rich, If I were to flip this coin a million times, it would come up heads at least once. If kangaroos lost their tails, they'd topple over.* The corresponding *strict* conditonals are not true, because it's *possible* that in some faraway world you woudn't be rich because you were already 300 million dollars in debt. But in nearby worlds, it seems, β holds if α does. First stab at the truth rule:

$\alpha > \beta$ is true at *w* iff $|\alpha| \subseteq |\beta|$ limiting ourselves to nearby worlds, that is, $|\alpha|\cap N \subseteq |\beta|\cap N$

Problem: "nearby" cannot be the same for every α, α may fail in all nearby worlds, making the conditional trivial. *If cats barked, they'd be scary.* Second try.

$\alpha > \beta$ is true at *w* iff nearby α-worlds are β-worlds.

Suppose a coin is flipped and I am offered a chance to bet on the outcome. I decline, but if I had bet, it would have been on heads. I want to say that *If I had bet, I would have won*, The coin could very easily have come up tails, though; including in nearby worlds where I bet. So there are nearby worlds where I bet but lost.

$\alpha > \beta$ is true at *w* iff the nearest α-world is a β-world. (Stalnaker)

What if there is no nearest $|\alpha|$-world? *If I you were over 5' tall, you could ride the roiler coaster.* There's no *closest* way of being over 5' tall, but still the conditional seems true. Why? Well, you can ride in all close-*enough* such worlds. Some *You're 5' tall and can ride the roller coaster*-worlds are closer than any *You're 5' tall and can't ride it*-worlds

$\alpha > \beta$ is true at *w* iff some *u* in $|\alpha| \cap |\beta|$ is more like *w* than any *v* in $|\alpha| \cap \overline{|\beta|}$ (Lewis)

Assess for validity, using Stalnaker's truth-conditions: the nearest α-world is a β-world.

If Hoover spoke Russian, he'd be bilingual....If she were in France, she wouldn't be in Paris....If Spain had fought with the Allies or the Axis....

Modus Ponens: $p \supset ((p \; \Box\!\!\rightarrow q) \supset q)$

Transitivity: $(p \; \Box\!\!\rightarrow q) \supset ((q \; \Box\!\!\rightarrow r) \supset (p \; \Box\!\!\rightarrow r))$

Contraposition: $(p \; \Box\!\!\rightarrow q) \supset (\neg q \; \Box\!\!\rightarrow \neg p)$

Simplification: $((p \vee q) \; \Box\!\!\rightarrow r) \supset (p \; \Box\!\!\rightarrow r)$

Probabilistic Semantics

The standard semantics for classical logic is two-valued: every sentence is evaluated either as true or false. What if we generalized that and allowed sentences to take arbitrary real values between 0 and 1?

Of course, you need a philosophical motivation for this. You might be interested in *vagueness* and an evaluation of 1/2 means that the sentence is midway between being true and false. If you are interested in *partial truth*, the 1/2 might mean that the sentence can be divided into two parts of which only one is true. *Snow is white and expensive* is a half-truth in that sense. Or, if you're interested in probability, the 1/2 signifies that there is half a chance in the speaker's mind of the sentence being true. The rules can in some cases be given exactly the same form: (we'll write P for V):

$P(\neg\alpha) = 1 - P(\alpha)$

But not always

??$P(\alpha \& \beta) = \min(P(\alpha), P(\beta))$??

??$P(\alpha \& \beta) = P(\alpha) \times P(\beta)$)??

$P(\alpha \& \beta) = P(\alpha) \times P(\beta)$) *provided they're independent*;

$P(\alpha \& \beta) = P(\alpha \mid \beta) \times V(\beta)$)—YES

Imagine $\beta = \neg\alpha$.

Imagine $\beta = \alpha$?

Yes but suppose they aren't.

A *Popper function* is any *binary* function P taking pairs of sentences to real numbers between 0 and 1 (inclusive), subject to six conditions

1. P (A|A)=1
2. P(B|A) is not always 1
3. if P(B|A)=1 and P(A|B)=1 then for all C, P(C|A)=P(C|B)
4. P(C&B|A) = P(C|B&A) × P(B|A)
5. P(C&B| A)<P(B|A)
6. P(¬B|A)=1-P(B|A), unless for all C, P(C|A)=1

Monadic probability is P(B) = P(B|T) where T is some fixed tautology.

The issue of how to probabilistic semantics for *classical* logic has been studied by Hartry Field in a paper called "Logic, Meaning, and Conceptual Role." Here we try to extend the idea to *modal* logic. Two strategies have been suggested, both employing conditional probabiliity but in quite different ways. The first understands worlds as hypotheses to be conditioned on, and truth in a world as high probability conditional on a hypothesis:

$P(\Box\alpha) = 1$ iff for all ω $P(\alpha|\omega) = 1$

The second treats the probability functions themselves as worlds. P' is accessible from P iff P' is obtainable from P by conditioning on some hypothesis *A*, that is, for all B and C, *P*'(C|B) = P(C|B&A). $P(\Box\alpha)$ is higher or lower depending on the probabilities assigned to α by probability functions P can see.

$P(\Box\alpha)$ = greatest lower bound of $P'(\alpha)$ as P' ranges over functions accessible from P.
$P(\Diamond\beta)$ = the least upper bound of $P'(\beta)$ as P' ranges over functions accessible from P.

The philosophical interpretation is not so clear to me. Imagine a thinker who's in a particular epistemic state but can imagine being in various other such states. If P represents the thinker's actual epistemic state, "the functions accessible to P represent those ... epistemic states that the agent... would recognize as distinct and intelligible alternatives" that she could reach by conditionalizing.

The meaning of $\Diamond\alpha$ is that one can imagine coming to think that α. Of course, the agent could just conditionalize P' on α to obtain a P" that made α certainly true. The problem is that this is not a possibility that our agent recognizes while in epistemic state P. She is supposedly unable to fill in the details of his thought experiment with sufficient specificity. Her confidence in $\Diamond\alpha$ corresponds the agent's ability to "fill in, with a sufficient degree of specificity, the details of a thought experiment" wherein it makes sense to believe that α. $P(\Diamond\alpha)$ will take an intermediate value if the agent can see her way through to epistemic states in which α is not certainly false but none in which α is certainly true.

Multi-Modal Logic

The language contains several $\Box$-operators $\Box_1$, $\Box_2$, etc. They are all defined over the same set of worlds but the accessibility relations are different. A model is <W,R,C> except that R is not a single accessibility relation but a list of them R_1, R_2, etc. Tense logic is one application; $\Box_1$ means it always will be the case, $\Box_2$ means it always has been the case; $\Diamond_1$ means it will sometime be the case and $\Diamond_2$ that it once was the case. R_1 is the relation of temporal precedence and R_2 the relation of coming after in time. Given that the relations are converses we can really get by with one of them in our truth rules; let it be the later-than relation.

$V(\Box_1\alpha,w) = 1$ iff $V(\alpha, u) = 1$ for all u later than w (wRu)
$V(\Box_2\alpha,w) = 1$ iff $V(\alpha, u) = 1$ for all u earlier than w. (uRw)

Both boxes appear to satisfy K:

If it will always be that $\alpha \supset \beta$, then if it will always be that α, it will always be that β
If it always was that $\alpha \supset \beta$, then if it always was that α, it always was that β

What about T?. What about S4 and S5? What about these two hybrid axioms linking the was and will-be operators? (Example adapted from Humphrey Bogart in Casablanca: "We'll always have Paris.")

TL1 If we have Paris now, then we will always have (had) Paris in the future. $p \supset \Box_1\Diamond_2 p$.
TL2 If we have Paris now, then we were always in the past going to have Paris. $p \supset \Box_2\Diamond_1 p$.

Is TL2 correct? Some say not on the theory that the past is fixed, the future open. Should there turn out to be a sea-battle tomorrow, does that make it true now that there will be a sea battle tomorrow?

So-called two-dimensionalism in the philosophy of language falls under this heading too. $\Box_1$ is regular old metaphysical necessity. $\Box_2$ is something like conceptual or epistemic necessity or a priority. *Julius invented the zip* is metaphysically contingent but conceptually necessary. Some popular glosses:

$\Box_1 p$ is true iff	$\Box_2 p$ is true iff
what p actually says had to be true	p had to expresses some truth or other
no matter which world *had* been actual, p	no matter which world *is* actual, p
p could not have failed to be true	p cannot fail to be actually true

The combination of these $\Box_1$ with @ greatly increases the language's expressive power, we'll see later. It comes out clearly in quantificational logic. *There could have been more stars than there are*. Adding $\Box_2$ in increases it further.

You snooze ⊃ You lose
You snooze ⊃ You snooze
You snooze ⊃ @(You lose)
@(You snooze) ⊃ You lose
@(You snooze) ⊃ @(You lose)

Necessity as a Predicate

Can we construct a theory $\mathcal{T}$ of *predicative* necessity? Let's agree that the theory has to include arithmetic and that the necessity predicate has to satisfy conditions A1-A4, the Montague conditions. $[A]$ is the Godel number of sentence A.

1. $\mathcal{T} \vdash \mathrm{N}([P]) \supset P$
2. $\mathcal{T} \vdash \mathrm{N}([\mathrm{N}([P]) \supset P])$
3. $\mathcal{T} \vdash \mathrm{N}([P \supset Q]) \supset (\mathrm{N}([P] \supset \mathrm{N}([Q]))$
4. $\mathcal{T} \vdash \mathrm{N}([P])$ if P is a truth of logic or arithmetic

Any such theory is bound to be inconsistent, Montague shows. He begins by letting λ be a "fixed point" of the formula $\neg\mathrm{N}(x$, meaning it's a theorem of arithmetic that $\lambda \equiv \neg\mathrm{N}([\lambda])$. All the numbered formulas are meant to be theorems of T.

1. $\lambda \supset \neg\mathrm{N}([\lambda])$ (theorem of arithmetic)
2. $\neg\mathrm{N}([\lambda]) \supset \lambda$ (theorem of arithmetic)
3. $\mathrm{N}([\lambda]) \supset \lambda$ (A1)
4. $(\mathrm{N}([\lambda]) \supset \lambda) \supset ((\lambda \supset \neg\mathrm{N}([\lambda])) \supset \neg\mathrm{N}([\lambda])$ (propositional logic)
5. $(\lambda \supset \neg\mathrm{N}([\lambda])) \supset \neg\mathrm{N}([\lambda])$ (modus ponens 3.4)
6. $\neg\mathrm{N}([\lambda])$ (modus ponens 1, 5)
7. λ (modus ponens 2, 6)
8. $\mathrm{N}([\lambda \supset \neg\mathrm{N}([\lambda])])$ (1, A4)
9. $\mathrm{N}([\neg\mathrm{N}([\lambda]) \supset \lambda])$ (2, A4)
10. $\mathrm{N}([\mathrm{N}([\lambda]) \supset \lambda])$ (A2)
11. $\mathrm{N}([(\mathrm{N}([\lambda]) \supset \lambda) \supset ((\lambda \supset \neg\mathrm{N}([\lambda])) \supset \neg\mathrm{N}([\lambda])])$ (4, A4)
12. $\mathrm{N}([\lambda \supset \neg\mathrm{N}([\lambda]) \supset \neg\mathrm{N}([\lambda])$ (A3 on 10,11)
13. $\mathrm{N}([\neg\ \mathrm{N}([\lambda])])$ (A3 on 8,12)
14. $\mathrm{N}([\lambda])$ (A3 on 9, 13)
15. $\neg\mathrm{N}([\lambda])$ (modus ponens on 1, 7)
16. $\mathrm{N}([\lambda])\ \&\ \neg\mathrm{N}([\lambda])$ (conjunction of14, 15)
17. !!!!!!!!!!!!!!!

Intuitively what's meant to be going is this. Assume for contradiction that λ is not true. Then what λ expresses is not the case, so λ is necessary. But then λ is true–since everything necessary is true—contradicting our assumption. This concludes the proof that λ is true. The proof is of a purely logical nature; so its conclusion is necessarily true, that is, $\mathrm{N}([\lambda])$. $\mathrm{N}([\lambda])$ is the negation of λ though, which was just proven. Contradiction.