

Philosophy 244: #10— Modal Predicate Metalogic

Soundness

Since the modal LPC models defined above all validate BF we'll speak of them as BF models. For the time being we'll be sticking to BF models, and defining validity in terms of these.

α is valid in BF model $\langle WRDV \rangle$ iff $V(\alpha, w) = 1$ for all worlds w .

α is valid on frame \mathcal{F} (= $\langle WR \rangle$) iff it is valid in every BF model based on \mathcal{F}

Our first result relates propositional modal validity to quantificational modal validity. It relies on a new notion: "frame for S."

A frame for logical system XX (e.g. S4)
= an \mathcal{F} such that each XX-theorem is valid on \mathcal{F} .

Prop. 13.1 If \mathcal{F} is a frame for normal modal system S, \mathcal{F} is a frame for S+BF.

Proof sketch: Let a frame \mathcal{F} for S be given. By definition each S-theorem is valid on \mathcal{F} . We need to show that each S+BF theorem is valid on \mathcal{F} , that is, valid in each BF model based on \mathcal{F} . ETS that each instance of axiom schemata S, $\forall 1$, and BF are valid on every BF model based on \mathcal{F} , and that the rules MP, NE, and $\forall 2$ preserve the property of being valid on every such model. Checking these things takes a while but is not hard. See pp.245-6 for details.

Prop. 13.2 If \mathcal{F} is a frame for S+BF, then \mathcal{F} is a frame for S.

Proof: Again, "tedious but straightforward." See pp. 248-9.

Cor. 13.3 \mathcal{F} is a frame for S iff it's a frame for S+BF.

From 13.1, it follows that every theorem of K+BF is valid on every frame whatsoever. We know that every frame whatsoever is a frame for K, ie., all propositional models based on these frames validate all K-theorems. 13.1 now tells us that all quantificational models based on these frames validate all theorems of S+BF. Sp S+BF is sound relative the class of all frames whatsoever. By the same style of argument, we can show that

T+BF	is sound over	reflexive frames
S4+BF	is sound over	reflexive, transitive frames
B+BF	is sound over	reflexive, symmetric frames
S5+BF	is sound over	reflexive, transitive, symmetric frames

Now, it might seem as though Prop. 13.2, the converse of 13.1, would give us similarly general *completeness* results. The reason it doesn't is kind of subtle. Take T + BF. We learn from 13.1, 13.2 that the frames for T+BF are precisely the reflexive frames. A frame for T+BF is a frame such that every theorem of T+BF is valid on it. So what we learn is

$$\{\mathcal{F} \mid \vdash_{T+BF} \alpha \Rightarrow \alpha \text{ is valid on } \mathcal{F}\} = \{\mathcal{F} \mid \mathcal{F} \text{ is reflexive}\}$$

What we want though, for soundness and completeness, is

$$\{\alpha \mid \mathcal{F} \text{ is reflexive} \Rightarrow \alpha \text{ is valid on } \mathcal{F}\} = \{\alpha \mid \vdash_{T+BF} \alpha\}$$

And that doesn't follow. In particular it doesn't follow from left to right, the completeness direction. For all we know so far, more formulas are valid on the reflexive frames than the theorems of T+BF. Maybe, that is,

This means essentially models with the same domain in every world; vary the domains and BF fails, e.g., $\Box \forall x \exists y y=x$ doesn't imply $\forall x \Box \exists y y=x$.

the reflexive frames are the only frames validating all of the theorems of T+BF, but the theorems of T+BF are not the only wffs validated by the reflexive frames.

Here is a dumb analogy:

TRUE: humans are the only animals that love their countries, but

FALSE: their countries are the only thing human animals worry about

This is not just an abstract possibility. Later we'll see that (although T is not one of them) there are systems S such that S is complete but S+BF is not.

De re and de dicto

The interest of quantified modal logic lies of course not in quantificational instances of propositional validities, but in new validities that crop up because of the way that operators and quantifiers interact. Take for instance

$$(1) \Box\exists x\varphi(x) \supset \exists x\Box\varphi(x)$$

which we discussed earlier. The converse of (1) is a theorem of any LPC+S but (1) itself is not. The interest of (1) lies in the fact that its antecedent has the quantifier within the modal operator, while its consequent has the operator within the quantifier. The antecedent is what we call a *de dicto* claim since all it is saying that a certain dictum of saying or thought is necessarily the case. The consequent is a *de re* claim since it is saying that a certain *res* (thing) necessarily has a certain property, the one expressed by φ .

DE DEF A wff α is

de dicto iff α has no variables appearing free within the scope of a modal operator.

de re iff α does have variables appearing free within the scope of a modal operator.

Time was when *de re* modal claims were under extreme suspicion, the leading skeptic being Quine. Quine was of the view that an object is necessarily thus and so only qua described in a certain way, not in itself. The view that objects in themselves have properties necessarily is

the kind of essentialism normally attributed to Aristotle, subject to correction by Aristotle scholars, such being the penalty for attributions to Aristotle

He can make no sense of essentialism. Essences for him are what meanings become when they are projected (illegitimately) from words onto objects. "Perhaps I can evoke the appropriate sense of bewilderment" by observing that 9 is necessarily greater than 7, while the number of planets only happens to be greater than 7, even though 9 = the number of planets. Mathematicians are necessarily rational accidentally bipedal, while bicyclists are necessarily bipedal and accidentally rational; and yet the very same person can be both a mathematician and a bicyclist.

Not every *de re* wff smacks of essentialism, even for Quine, for instance, $\forall x\Box(Px \supset Px)$. A *de re* wff is presumably OK if it is equivalent to a *de dicto* wff. The question arises as to whether there are modal systems in which *all* *de re* wffs are equivalent to *de dicto* wffs. These systems and presumably only these would be acceptable to antiessentialists.

No such systems exist, however. The proof relies on an interesting fact about *de dicto* wffs. They do not depend for their truth-value on how individuals match up across distinct worlds ("transworld heir lines"). Example: Suppose we have two models $\langle WRDV \rangle$ and $\langle WRDV^* \rangle$. Both have two worlds w_1 and w_2 , a universal accessibility relation, and $D = \{a_1, a_2\}$. They differ in that V maps every predicate other than φ to the null set, and φ to

He didn't consider himself a leading logician but only "captain of the B team."

Nowadays these are apt to strike us as scope confusions. Arthur Smullyan pointed this out already in 1947 in "Modality and Descriptions." But Quine doesn't deny that essentialism is intelligible, he just think it's nuts, because of being metaphysical rather than semantic. Analytic necessity is *de dicto* he assumes. Fine disagrees. He develops a notion of analytic *de re* necessity in "The Problem of De Re Modality."

$\langle a_1, w_1 \rangle, \langle a_1, w_2 \rangle$. V^* is the same except that $V^*(\varphi)$ is $\langle a_1, w_1 \rangle, \langle a_2, w_2 \rangle$. Whichever model you pick, a_1 is φ in w_1 , but in the second the φ switches to a_2 in w_2 . The idea behind the proof is that if α is de dicto, it shouldn't care that the second model has switched a_1 and a_2 in w_2 . Both models are still made up of worlds in which one of the two things is φ , in each world. If α is de re, however, the crossworld relations may matter.

Some terminology: the anti-assignment μ^* to μ is the unique ρ which differs from μ only in assigning to x the "other" thing, the thing not identical to $\mu(x)$ (we're sticking here to the two-element domain).

Prop. 13.4 If α is de dicto then $V_\mu(\alpha, w_1) = V_{\mu^*}(\alpha, w_1)$ and $V_\mu(\alpha, w_2) = V_{\mu^*}(\alpha, w_2)$

$V_{\mu^*}(\alpha, w_2) = V_\mu(\alpha, w_2)$ since α has no free variables.

Proof by induction on complexity of α .

Now we show that $\exists x \Box \varphi(x)$ is not equivalent in $S5+BF$ to any de dicto wff. 13.3 assures us that \mathcal{M} and \mathcal{M}^* both satisfy all theorems of $S5+BF$. Suppose for contradiction a de dicto wff α exists such that $\vdash_{S5+BF} \exists x \Box \varphi(x) \equiv \alpha$. Then for every μ and w .

$$\begin{aligned} V_\mu(\exists x \Box \varphi(x), w) &= V_\mu(\alpha, w), \\ V_{\mu^*}(\exists x \Box \varphi(x), w) &= V_{\mu^*}(\alpha, w). \end{aligned}$$

Now, α is de dicto so by 13.4, $V_\mu(\alpha, w_1) = V_{\mu^*}(\alpha, w_1)$. But then, α does not agree with $\exists x \Box \varphi(x)$ in w_1 unless

$$V_\mu(\exists x \Box \varphi(x), w_1) = V_{\mu^*}(\exists x \Box \varphi(x), w_1).$$

$V_\mu(\exists x \Box \varphi(x), w_1)$ is 1, because a_1 is φ in both worlds (by the lights of V). $V_{\mu^*}(\exists x \Box \varphi(x), w) = 0$, because nothing is φ in both worlds (by the lights of V^*).

If the model had more elements, the role of the antiassignment μ^* would be played by a permutation π of the domain. Kit Fine has shown that not only do de dicto formulas retain their truth-values through permutations, but they are the only formulas with this property (up to equivalence, that is, the permutation invariant wffs are either de dicto or equivalent to so something de dicto). *So permutation-invariance is the semantic equivalent of the syntactic notion of de re defined above.*

Model Theory for Modal Logic, Part I: The De Re-De Dicto Distinction. He suggests if I remember right a mini-theory of aboutness: a wff is not "about" any particular object iff its truth-value can never be changed by a permutation of the domain

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