# *Philosophy 244: #13— Shifting Domains*

The BF has been shown a lot of deference so far. There is such a thing as life without it. LPC+S is one thing, LPC+S+BF (which we've been calling S+BF) another. The one has two axioms:

S' $\vdash\alpha$ for each $\alpha$ an LPC substitution instance of an S-theorem,
∀1 $\vdash\forall x\alpha\supset\alpha[y/x]$

And three rules

NE $\vdash\alpha \supset \vdash\Box\alpha$
MP $\vdash\alpha$, $\vdash\alpha\supset\beta \Rightarrow \vdash\beta$
∀2 $\vdash\alpha\supset\beta \supset \vdash \alpha\supset\forall x\beta$ provided *x* is not free in $\alpha$

The other has all of this plus an additional axiom

BF $\vdash\forall x\Box\alpha\supset\Box\forall x\alpha$

Recall BF is not *in general* a theorem of LPC+S; it is not for instance when S is K, or D, or T, or S4.

## *Expanding Domains*

Now, the models we've been looking at so far — models in which every world has the same domain — validate the Barcan Formula. So, if we want to do semantics for systems like LPC+K that don't have BF as a theorem, we'll have to move to a more general kind of model. And in fact we do want to do semantics for systems like that, because BF is in many contexts intuitively objectionable. It says that if everything is necessarily made of atoms, then necessarily everything is made of atoms. And that's just not a logical truth. There could have been a plenum world.

A cleaner example is this. Spose that $a_1$, $a_2$,...., $a_n$ is a list of all the existing things. Then to go by BF, for all *x*, it's necessary that *x* is $a_1$, $a_2$,...or $a_n$. But it's not necessary that all the *x*s are one of these $a_i$s. For it could have been that there was an additional thing!

This suggests that to accommodate LPC+S without BF, we need a semantics in which world *w* with its inhabitants can see worlds with all those inhabitants and more. It is not required, for these purposes, that we admit unrestricted variation in domain, just that we allow domains to expand. Here's the definition.

> A model (until further notice) is a quintuple <WRDQV>, where WRD are as before, and Q is a function from worlds to subsets of D — Q(*w*) = $D_w$ — such that *w*R*w*' only if $D_w\subseteq D_{w'}$.

And what about V? How should $\varphi(x)$ be evaluated in a world *w* such that the object assigned to *x* does not exist in *w*? There are three options:

Avoidance: Prohibit such assignments.

Undefined: $V_\mu(\varphi(x),w)$ is undefined; $\varphi(x)$ is neither true nor false
Defined: $V_\mu(\varphi(x),w)$ is defined; $\varphi(x)$ is true or false as usual.

This last is the option we'll take. *Socrates is a philosopher* gets a truth-value not only in worlds where he exists but also in worlds where he doesn't. Not to worry, though; the second and third options yield the same results as regards validity. The effect on the evaluation rules is that all the rules remain exactly what they were except that now we have:

($V\forall$) $V_\mu(\forall x\alpha,w)=1$ iff $V_\rho(\alpha,w)=1$ for every $x$-alternative $\rho$ of $\mu$ s.t. $\rho(x)\epsilon D_w$.

That is, what $\forall x\alpha$ says about world $w$ is that all the $w$-things – all the things in $D_w$ – are $\alpha$. Other things may be non-$\alpha$ in $w$ but that's OK, provided they don't exist there. (E.g., *Nothing is human* may be true in a world where Socrates is human, provided he doesn't belong to the world's domain.)

$\alpha$ is valid in <WRDQV> iff for each $w\epsilon W$,
$V_\mu(\alpha,w)=1$ for each $\mu$ mapping each $x$ to a member of $D_w$.

An equivalent definition is this. Say that $\alpha$ is *eligible* at $w$ w.r.t. $\mu$ iff $\mu(x)\epsilon D_w$ for every $x$ free in $\alpha$. $\alpha$ is valid in <WRDQV> iff it is true at every world where it is eligible.

As before, an inductive argument shows that any model that validates the theorems of S validates those of LPC+S. To show that BF is not a theorem of LPC+S, then, it suffices to show that some S-model fails to validate BF. Try it for S=K. Our model is

$W = \{w_1, w_2\}$,
$w_1 R\, w_2$,
$D_{w1} = \{u_1\}$, $D_{w2} = \{u_1, u_2\}$.
$V(\varphi) = \{< u_1, w_1>,< u_1, w_2>\}$.

How does $\Box\forall x\varphi x\supset\forall x\Box\varphi x$ fare in this model? Is BF a theorem of LPC+K? Can we similarly show BF is not a theorem of LPC+T or LPC+S4?

But consider LPC+B. The given model isn't symmetrical, so the fact that it invalidates BF doesn't show that BF isn't a theorem. Moreover if you try to *make* the model symmetrical by letting $w_2$ see $w_1$, you violate the expansion requirement. Moreover if you try to fix this by letting $u_2$ into $w_1$, BF is no longer false in $w_1$. So it so far left open that BF may be a theorem of LPC+B.

## *Truth-Value Gaps*

Back to the issue of how $\varphi(x)$ should be evaluated in a world $w$ where its value under $\mu$ does not exist. We picked the third option, but suppose we'd tried the second instead: $\varphi x$ is undefined in such a world. Then a question arises: how does $\varphi x$ contribute to the truth-value of larger formulas in which it figures? One obvious thought is the weak Kleene scheme, which passes gappiness in a component up to the whole.

Weak Kleene finds F&N and T$\vee$N to be gappy. Strong Kleene finds the first false and the second true.

"A jot of rat dung spoils the soup."

($V^*\varphi$) $V^*_\mu(\varphi(x_1..x_n),w)=1$ (0) iff each $\mu(x_i)\epsilon D_w$ & $<\mu(x_1)...\mu(x_n),w>\epsilon V^*(\varphi)$ ($<\mu(x_1)..\mu(x_n),w>\notin V^*(\varphi)$)

($V^*\neg$) $V^*_\mu(\neg\alpha,w) = 1$ (0) iff $V^*_\mu(\alpha,w)=0$ (1)

($V^*\vee$) $V^*_\mu(\alpha\vee\beta,w) = 1$ (0) iff *both are defined* and either is 1 (both are 0).

($V^*\Box$) $V^*_\mu(\Box\alpha,w) = 1$ (0) iff $V^*_\mu(\alpha,w')$ is defined on each $w'\epsilon R(w)$ and (not) always 1

($V^*\forall$) $V^*_\mu(\forall x\alpha,w)=1$ (0) iff $V^*_\rho(\alpha,w)$ is (not) 1 for each $x$-alternative $\rho$ s.t. $\rho(x)\epsilon D_w$

$\alpha$ is valid* iff for every $w$ and every $\mu$, $V_\mu{}^*(\alpha,w)=1$ whenever it is defined.

Prop. 15.1 If $\mu(x)\epsilon D_w$ for every $x$ free in $\alpha$, then $V_\mu{}^*(\alpha,w) = V_\mu(\alpha,w)$.

The book doesn't list it as a separate result, but the reason 15.1 matters is this corollary: $\alpha$ is valid iff $\alpha$ is valid*.

## *Canonical Models*

The key is again to construct a canonical model—except now the domain can't be (just) the set of variables because such a domain will be constant across worlds.

As before we work with an original language $\mathcal{L}$ and an extended one $\mathcal{L}+$ with infinitely many new variables. Each world $w$ will be a maximal consistent set in a language $\mathcal{L}_w$ which contains all the variables of $\mathcal{L}$ and (possibly) some of those in $\mathcal{L}+$, and which is such that there are infinitely many variables left over. Based on just that we can prove:

Def. An *infinitely proper subset* of Y iff is a subset falling short of Y by infinitely many members. Languages are related same way.

Prop. 15.2 If $\neg\Box\alpha\epsilon w$, then there is a maxiset $w'$ with the $\forall$-property in a language $\mathcal{L}_{w'}$ containing $\mathcal{L}_w$ such that $\Box^-(w)\cup\{\neg\alpha\}\subseteq w'$.

Lemma 6.4: Let S be a normal system of PML; if $\Lambda$ is S-consistent with $\neg\Box\alpha$. Then $\Box^-(\Lambda)$ is S-consistent with $\neg\alpha$.

Proof. Let $\mathcal{L}_{w'}$ be (both) an infinitely proper sublanguage of $\mathcal{L}+$ and superlanguage of $\mathcal{L}_w$. By definition $\mathcal{L}_w$ falls short of $\mathcal{L}+$ by infinitely many variables. $\Box^-(w)\cup\{\neg\alpha\}$ is a set of $\mathcal{L}_w$ sentences that is consistent by lemma 6.4 on p. 117. Since $\mathcal{L}_{w'}$ contains infinitely many variables not in $\mathcal{L}_w$, 14.1 guarantees that $\Box^-(w)\cup\{\neg\alpha\}$ has a consistent extension $\Delta$ with the $\forall$-property in $\mathcal{L}_{w'}$. Prop. 6.3 on p. 115 tells us this can be extended to a maxiset $w'$. □

Theorem 14.1: Any consistent set of $\mathcal{L}$-wff can be extended to a consistent set of wff of $\mathcal{L}+$ with ther $\forall$-property.

So: the canonical model of LPC+S in a language $\mathcal{L}$ with extension $\mathcal{L}+$ is <WRDQV>, where the five components are defined as follows:

W = the set of all maxisets-with-the-$\forall$-propty of some proper sublanguage of $\mathcal{L}+$
$w$R$w'$ iff $\Box\alpha\epsilon w$ only if $\alpha\epsilon w'$;
D is the set of variables of $\mathcal{L}+$,
$Q(w)=D_w$ = the set of variables in $\mathcal{L}_w$; and
$<x_1,...x_n,w>\epsilon V(\varphi)$ iff $\varphi(x_1....x_n)\epsilon w$.

Theorem 6.3: Any consistent set of wff can be extended to a maximal consistent set of wff.

Let's check that the expansion condition is satisfied because that's what invalidates the Barcan Formula. Suppose $x\epsilon D_w$. Then $\Box(\varphi(x)\supset\varphi(x))\epsilon w$, so for any visible $w'$ $\varphi(x)\supset\varphi(x)\epsilon w'$; so $x\epsilon Dw'$ for every $w'$ visible from $w$.

If we let the canonical variable assignment $\sigma$ be the identity function— $\sigma(x)=x$—as before, we can prove by induction on complexity that

Prop. 15.3 $\forall w\epsilon W$, and $\alpha\epsilon\mathcal{L}_w$, $V_\sigma(\alpha,w)=1$ iff $\alpha\epsilon w$.

The canonical model is an unlovely thing which we care about only because of its relation to

## *Completeness*

From 15.3 it is clear that $\alpha$ is valid in $\mathcal{M}_{S+BF}$ iff $\vdash_{S+BF} \alpha$. Completeness follows if we can show that the frame of $\mathcal{M}_{S+BF}$ is a frame for S. One can prove completeness for LPC+K, D, T, and S4 in this way.

Won't review this again.

But not B! Why? Accessibility was defined so that *w*R*w*' only if the language of *w* is a sublanguage of that of *w*'. Often it will be a proper sublanguage. But then R can't be symmetrical because you'd need $\mathcal{L}_{w'}$ to be a proper sublanguage of $\mathcal{L}w$ as well. Because the canonical model for LPC+B is not symmetrical, its frame will not be among the B-frames and hence our method of proving completeness won't work.

BUT: we saw in the last chapter that BF is a theorem of LPC+B has BF has a theorem. So we're really dealing with S+BF when S=B. And so the method of the last chapter will work; you can stick to models with constant domains. But there are systems for which neither of the two methods will work. You can't use constant domains because BF is not a theorem; you can't use expanding domains because at least some pairs of worlds have got to see each other. The book gives K + $\Diamond\Box(p\&\neg p)\vee(q\supset\Box\Diamond q)$ as an example. See 282-3.

## *Shrinking Domains and Converse Barcan*

A model will validate BF if the domains are the same, or smaller, in accessible worlds. It will validate ¬BF if domains get bigger, or become incomparable, as you move from worlds to accessible worlds. But now what about

CBF $\Box\forall x\alpha\supset\forall x\Box\alpha$.

CBF seems at first more attractive. . If $\alpha$ expresses any "ordinary" property, e.g., being concrete or human, then CBF is just right. The only time it lets us down is when $\alpha$ is an existence-involving property, for instance, existence itself. Maybe we should look then at models with expanding domains? No, because the existence case is bad enough. CBF regarding existence says that everything has a property usually reserved to abstract objects or divinities.

## *Variable Domains and Existence*

So, time has come to abandon the inclusion requirement in either direction. This has some initially odd results. One is that necessitation no longer preserves validity. $\forall x\varphi(x)\supset\varphi y$ is valid (it's axiom ∀1), but its necessitation isn't.

Spose $D_w = \{u_1, u_2\}$, and $D_{w'} = \{u_1\}$. And let $\varphi$ apply to both *u*s in *w* but only to $u_1$ in *w*'. Then assuming that $\mu$(y)= $u_2$, $V_\mu(\forall x\varphi(x)\supset\varphi(y),w')=0$ , so $V_\mu(\Box(\forall x\varphi(x)\supset\varphi(y)),w)=0$, so $\Box(\forall x\varphi(x)\supset\varphi(y))$ isn't valid.

What to do? Maybe ∀1 only seemed valid because there was a valid formula in the neighborhood: $\forall x\varphi(x)\supset(E(y)\supset\varphi(y))$. The hedged version can be necessitated. To formulate it though we need an existence predicate. Which leads to our next topic: the proper treatment of existence in a modal language.

Kripke noticed early on that you have to be careful about necessitation if you allow free variables in theorems. From $\alpha x$) you would be able to get $\forall x\Box\alpha(x)$. Whereas from $\forall x\ \alpha(x)$ nothing like that follows. Weird that you should be able to deduce more from the instance than the universal generalization.

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