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21L.017 The Art of the Probable: Literature and Probability
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THE DOCTRINE OF CHANCES: or
A METHOD OF CALCULATING THE PROBABILITIES OF EVENTS IN PLAY
by A. DE MOIVRE (3rd edition, 1756)

[....]

Such a Tract as this is may be useful to several ends; the first of which is, that there being in the World several inquisitive Persons, who are desirous to know what foundation they go upon, when they engage in Play, whether from a motive of Gain, or barely Diversion, they may, by the help of this or the like Tract, gratify their curiosity, either by taking the pains to understand what is here Demonstrated, or else making use of the Conclusions, and taking it for granted that the Demonstrations are right.

Another use to be made of this Doctrine of Chances is, that it may server in Conjunction with the other parts of the Mathematics, as a fit Introduction to the Art of Reasoning; it being known by experience that nothing can contribute more to the attaining of that Art, than the consideration of a long Train of Consequences, rightly deduced from undoubted Principles, of which this Book affords many Examples. To this may be added, that some of the Problems about Chance having a great appearance of Simplicity, the Mind is easily drawn into a belief, that their Solution may be attained by the mere Strength of natural good Sense; which generally proving otherwise, and the Mistakes occasioned thereby being not unfrequent, 'tis presumed that a Book of this Kind, which teaches to distinguish Truth from what seems so nearly to resemble it, will be looked upon as a help to good Reasoning.

Among the several Mistakes that are committed about Chance, one of the most common and least suspected, is that which relates to Lotteries. Thus, supposing a Lottery wherein the proportion of the Blanks to the Prizes is as five to one; 'tis very natural to conclude, that therefore five Tickets are requisite for the Chance of a Prize; and yet it may be proved, Demonstratively, that four Tickets are more than sufficient for that purpose, which will be confirmed by often repeated Experience. In the like manner, supposing a Lottery wherein the proportion of the Blanks to the Prizes is a Thirty-nine to One, (such as was the Lottery of 1710) it may be proved, that in twenty eight Tickets, a Prize is as likely to be taken as not; which tho' it may seem to contradict the common Notions, is nevertheless grounded upon infallible Demonstration.

When the Play of the Royal Oak was in use, some Persons who lost considerably by it, had their Losses chiefly occasioned by a Argument of which they could not perceive the Fallacy. The Odds against any particular Point of the Ball were One and Thirty to One, which entitled the Adventurers, in case they were winners, to have thirty two Stakes returned, including their own; instead of which they having but Eight and Twenty, it was very plain that on the single account of the disadvantage of the Play, they lost one eighth part of all the Money they played for. But the Master of the Ball maintained that they had no reason to complain; since he would undertake that any particular point of the Ball should come up in Two and Twenty Throws; of this he would offer to lay a Wager, and actually laid it when required. The seeming contradiction between the Odds of One and Thirty to One, and Twenty-two Throws for any Chance to come up, so perplexed the Adventurers, that they begun to think the Advantage was on their side; for which reason they played on and continued to lose.

The Doctrine of Chances may likewise be a help to cure a Kind of Superstition, which has been of long standing in the World, viz. that there is in Play such a thing as Luck, good or bad. I own there are a great many judicious people, who without any other Assistance than that of their own reason, are satisfied, that the Notion of Luck is merely Chimerical; yet I conceive that the ground they have to look upon it as such,

may still be farther enforced from some of the following Considerations. If by saying that a Man has had good Luck, nothing more was meant than that he has been generally a Gainer at play, the Expression might be allowed as very proper in a short way of speaking: But if the Word Good Luck be understood to signify a certain predominant quality, so inherent in a Man, that he must win whenever he Plays, or at least win oftener than lose, it may be denied that there is any such thing in nature.

The Asserters of Luck are very sure from their own Experience, that at some times they have been very Lucky, and that at other times they have had a prodigious Run of ill Luck against them, which whilst it continued obliged them to be very cautious in engaging with the Fortunate; but how Chance should produce those extraordinary Events, is what they cannot conceive: They would be glad, for Instance, to be Satisfied, how they could lose Fifteen Games together at Piquet, if ill Luck had not strangely prevailed against them. But if they will be pleased to consider the Rules delivered in this Book, they will see, that though the Odds against their losing so many times together be very great, viz. 32767 to 1, yet that the Possibility of it is not destroyed by the greatness of the Odds, there being One Chance in 32768 that it may so happen; from whence it follows, that it was still possible to come to pass without the Intervention of what they call Ill Luck.

Besides, This Accident of losing Fifteen times together at Piquet, is no more to be imputed to ill Luck, than the Winning with one single Ticket the highest Prize, in a Lottery of 32768 Tickets, is to be imputed to good Luck, since the Chances in both Cases are perfectly equal. But if it be said that Luck has been concerned in this latter Case, the Answer will be easy; for let us suppose Luck not existing, or at least let us suppose its influence to be suspended, yet the highest Prize must fall into some Hand or other, not by Luck, (for by the Hypothesis that has been laid aside) but from the mere necessity of its falling somewhere.

Those who contend for Luck, may, if they please, allege other Cases at Play, much more unlikely to happen than the Winning or Losing fifteen Games together, yet still their Opinion will never receive any Addition of Strength from such Suppositions: For, by the Rules of Chance, a time may be computed, in which those Cases may as probably happen as not; nay, not only so, but a time may be computed in which there may be any proportion of Odds for their so happening. But supposing that Gain and Loss were so fluctuating, as always to be distributed equally, whereby Luck would certainly be annihilated; would it be reasonable in this Case to attribute the Events of Play to Chance alone? I think, on the contrary, it would be quite otherwise, for then there would be more reason to suspect that some unaccountable Fatality did rule in it: Thus, if two Persons play at Cross and Pile, and Chance alone be supposed to be concerned in regulating the fall of the Piece, is it probable that there should be an Equality of Heads and Crosses? It is Five to Three that in four times there will be an inequality; 'tis Eleven to Five in six, 93 to 35 in Eight, and about 12 to 1 in a hundred times: Wherefore Chance alone by its Nature constitutes the Inequalities of Play, and there is no need to have recourse to Luck to explain them.

Further, the same Arguments which explode the Notion of Luck, may, on the other side, be useful in some Cases to establish a due comparison between Chance and Design: We may imagine Chance and Design to be, as it were, in Competition with each other, for the production of some sorts of Events, and may calculate what Probability there is, that those Events should be rather owing to one than to the other. To give a familiar Instance of this, Let us suppose that two Packs of Piquet- Cards being sent for, it should be perceived that there is, from Top to Bottom, the same Disposition of the Cards in both Packs; let us likewise suppose that, some doubt arising about this Disposition of the Cards, it should be questioned whether it ought to be attributed to Chance, or to the Maker's Design: In this Case the Doctrine of Combinations decides the Question; since it may be proved by its Rules, that there are the Odds of above

263130830000 Millions of Millions of Millions of Millions to One, that the Cards were designedly set in the Order in which they were found.

From this last Consideration we may learn, in many Cases, how to distinguish the Events which are the effect of Chance, from those which are produced by Design: The very Doctrine that finds Chance where it really is, being able to prove by a gradual Increase of Probability, till it arrive at Demonstration, that where Uniformity, Order and Constancy reside, there also reside Choice and Design.

Lastly, One of the principal Uses to which this Doctrine of Chances may be applied, is the discovering of some Truths, which cannot fail of pleasing the Mind, by their Generality and Simplicity; the admirable Connection of its Consequences will increase the Pleasure of the Discovery; and the seeming Paradoxes wherewith it abounds, will afford very great matter of Surprise and Entertainment to the Inquisitive. A very remarkable Instance of this nature may be seen in the prodigious Advantage which the repetition of Odds will amount to; Thus, Supposing I play with an Adversary who allows me the Odds of 43 to 40, and agrees with me to play till 100 Stakes are won or lost on either side, on condition that I give him an Equivalent for the Gain I am entitled to by the Advantage of my Odds; the Question is, what I am to give him, on supposing we play a Guinea a Stake: The Answer is 99 Guineas and above 18 Shillings, which will seem almost incredible, considering the smallness of the Odds of 43 to 40. Now let the Odds be in any Proportion given, and let the Number of Stakes be played for be never so great, yet one general Conclusion [i.e., formula] will include all the possible Cases, and the application of it to Numbers may be wrought [i.e., worked out] in less than a Minute's time.

[...]

THE INTRODUCTION

1. The Probability of an Event is greater or less, according to the number of Chances by which it may happen, compared with the whole number of Chances by which it may either happen or fail.
2. Wherefore, if we constitute a Fraction whereof the Numerator be the number of Chances whereby an Event may happen, and the Denominator the number of all the Chances whereby it may either happen or fail, that Fraction will be a proper designation of the Probability of happening. Thus if an Event has 3 Chances to happen, and 2 to fail, the Fraction $\frac{3}{5}$ will fitly represent the Probability of its happening, and may be taken to be the measure of it. The same thing may be said of the Probability of failing, which will likewise be measured by a Fraction whose Numerator is the number of Chances whereby it may fail, and the Denominator the whole number of Chances, both for its happening and failing; thus the Probability of the failing of that Event which has 2 Chances to fail and 3 to happen will be measured by the Fraction $\frac{2}{5}$.
3. The Fractions which represent the Probabilities of happening and failing, being added together, their Sum will always be equal to Unity, since the Sum of their Numerators will be equal to their common Denominator: now it being a certainty that an Event will either happen or fail, it follows that Certainty, which may be conceived under the notion of an infinitely great degree of Probability, is fitly represented by Unity.

These things will easily be apprehended, if it be considered, that the word Probability includes a double Idea; _first, of the number of Chances whereby an Event may happen; secondly, of the number of Chances whereby it may either happen or fail.

If I say that I have three Chances to win any Sum of Money, it is impossible from that bare assertion to judge whether I am like to obtain it; but if I add that the number of Chances either to obtain it, or to miss it, is five in all, from hence will ensue a comparison between the Chances that favor me, and the whole number of Chances that are for or against me, whereby a true judgment will be formed of my Probability of success: from whence it necessarily follows, that it is the comparative magnitude of the number of Chances to happen, in respect to the whole number of Chances either to happen or to fail, which is the true measure of Probability.

4. If upon the happening of an Event, I be entitled to a Sum of Money, my Expectation of obtaining that Sum has a determinate value before the happening of the Event.

Thus, if I am to have 10£: in case of the happening of an Event which has an equal Probability of happening and failing, my Expectation before the happening of the Event is worth 5£: for I am precisely in the same circumstances as he who at an equal Play ventures 5£: either to have 10, or to lose his 5. Now he who ventures 5£: at an equal Play, is possessor of 5£: before the decision of the Play; therefore my Expectation in the case above-mentioned must also be worth 5£.

5. In all cases, the Expectation of obtaining any Sum is estimated by multiplying the value of the Sum expected by the Fraction which represents the Probability of obtaining it. Thus, if I have 3 Chances in 5 to obtain 100£: I say that the present value of my Expectation is the product of 100£: by the fraction $\frac{3}{5}$, and consequently that my expectation is worth 60£: For supposing that an Event may equally happen to any one of 5 different Persons, and that the Person to whom it happens should in consequence of it obtain the Sum of 100£: it is plain that the right which each of them in particular has upon the Sum expected is $\frac{1}{5}$ of 100£: which right is founded in this, that if the five Persons concerned in the happening of the Event, should agree not to stand the Chance of it, but to divide the Sum expected among themselves, then each of them must have $\frac{1}{5}$ of 100£: for his pretension. Now whether they agree to divide that sum equally among themselves, or rather chose to stand the Chance of the Event, no one has thereby any advantage or disadvantage, since they are all upon an equal foot, and consequently each Person's expectation is worth $\frac{1}{5}$ of 100£: Let us suppose farther, that two of the five Persons concerned in the happening of the Event, should be willing to resign their Chance to one of the other three; then the Person to whom those two Chances are thus resigned has now three Chances that favor him, and consequently has now a right triple of that which he had before, and therefore his expectation is now worth $\frac{3}{5}$ of 100£: Now if we consider that the fraction $\frac{3}{5}$ expresses the Probability of obtaining the Sum of 100£:, and that $\frac{3}{5}$ of 100, is the same thing as $\frac{3}{5}$ multiplied by 100, we must naturally fall into this conclusion, which has been laid down as a principle, that the value of the Expectation of any Sum, is determined by multiplying the Sum expected by the Probability of obtaining it.

This manner of reasoning, tho' deduced from a particular case, will easily be perceived to be general, and applicable to any other case.

Corollary.

From what precedes, it necessarily follows that if the Value of an Expectation be given, as also the Value of the thing expected, then dividing the first value by the second, the quotient will express the Probability of obtaining the Sum expected: thus if I have an Expectation worth 60£: and that the Sum which I may obtain be worth 100£: the Probability of obtaining it will be expressed by the quotient of 60 divided by 100, that is by the fraction $\frac{60}{100}$ or $\frac{3}{5}$.

6. The Risk of losing any Sum is the reverse of Expectation; and the true measure of it is, the product of the Sum adventured multiplied by the Probability of the Loss.

7. Advantage or Disadvantage in Play, results from the combination of the several Expectations of the Gamesters, and of their several Risks. Thus supposing that A and B play together, that A has deposited 5L and B 3L: that the number of Chances which A has to win is 4, and the number of Chances which B has to win is 2, and that it were required in this circumstance to determine the advantage or disadvantage of the Adventurers, we may reason in this manner: Since the whole Sum deposited is 8, and that the Probability which A has of getting it is $\frac{4}{6}$, it follows that the Expectation of A upon the whole Sum deposited is $8 \cdot \frac{4}{6} = 5 \frac{1}{3}$, and for the same reason the Expectation of B upon that whole Sum deposited is $8 \cdot \frac{2}{6} = 2 \frac{2}{3}$.

Now, if from the respective Expectations which the Adventurers have upon the whole sum deposited, be subtracted the particular Sums which they deposit, that is their own Stakes, there will remain the Advantage or Disadvantage of either, according as the difference is positive or negative.

And therefore if from $5 \frac{1}{3}$, which is the Expectation of A upon the whole Sum deposited, 5 which is his own Stake, be subtracted, there will remain $\frac{1}{3}$ for his advantage; likewise if from $2 \frac{2}{3}$ which is the Expectation of B, 3 which is his own Stake be subtracted, there will remain $-\frac{1}{3}$, which being negative shows that his Disadvantage is $\frac{1}{3}$.

These conclusions may also be derived from another consideration; for if from the Expectation which either Adventurer has upon the Sum deposited by his Adversary, be subtracted the Risk of what he himself deposits, there will likewise remain his Advantage or Disadvantage, according as the difference is positive or negative.

Thus in the preceding case, the Stake of B being 3, and the Probability which A has of winning it, being $\frac{4}{6}$, the Expectation of A upon that Stake is $3 \cdot (\frac{4}{6}) = 2$; moreover the Stake of A being 5, and the Probability of losing it, being $\frac{2}{6}$, his Risk ought to be estimated by $5 \cdot (\frac{2}{6}) = 1 \frac{2}{3}$; wherefore, if from the Expectation 2, the Risk $1 \frac{2}{3}$ be subtracted, there will remain $\frac{1}{3}$ as before for the Advantage of A: and by the same way of proceeding, the Disadvantage of B will be found to be $\frac{1}{3}$.

It is very carefully to be observed, that what is here called Advantage or Disadvantage, and which may properly be called Gain or Loss, is always estimated before the Event is come to pass; and altho' it be not customary to call that Gain or Loss which is to be derived from an Event not yet determined, nevertheless in the Doctrine of Chances, that appellation is equivalent to what in common discourse is called Gain or Loss.

For in the same manner as he who ventures a Guinea in an equal Game may, before the determination of the Play, be said to be possessor of that Guinea, and may, in consideration of that Sum, resign his place to another; so he may be said to be a Gainer or Loser, who would get some Profit, or suffer some Loss, if he would sell his Expectation upon equitable terms, and secure his own Stake for a Sum equal to the Risk of losing it.

8. If the obtaining of any Sum requires the happening of several Events that are independent on each other, then the Value of the Expectation of that Sum is found by multiplying together the several Probabilities of happening, and again multiplying the product by the Value of the Sum expected.

Thus supposing that in order to obtain 90£ two Events must happen; the first whereof has 3 Chances to happen, and 2 to fail, the second has 4 Chances to happen, and 5 to fail, and I would know the value of that Expectation; I say, The Probability of the first's happening is $\frac{3}{5}$, the Probability of the second's happening is $\frac{4}{9}$; now multiplying these two Probabilities together, the product will be $\frac{12}{45}$ or $\frac{4}{15}$; and this product being again multiplied by 90, the new product will be $\frac{360}{15}$ or 24, therefore that Expectation is worth 24£:

The Demonstration of this will be very easy, if it be consider'd, that supposing the first Event had happened, then that Expectation depending now entirely upon the second, would, before the determination of the second, be found to be exactly worth $\frac{4}{9} \times 90$ £: or 40£: (by Art. 5th) We may therefore look upon the happening of the first, as a condition of obtaining an Expectation worth 40£: but the Probability of the first's happening has been supposed $\frac{3}{5}$, wherefore the Expectation sought for is to be estimated by $\frac{3}{5} \times 40$, or $\frac{3}{5} \times (\frac{4}{9} \times 90)$; that is, by the product of the two Probabilities of happening multiplied by the Sum expected.

And likewise, if an Expectation depends on the happening of one Event, and the failing of another, then its Value will be the product of the Probability of the first's happening by the Probability of the second's failing, and of that again by the Value of the Sum expected.

4 The Doctrine of Chances. And again, if an Expectation depends on the failing of two Events, the Rule will be the same; for that Expectation will be found by multiplying together the two Probabilities of failing, and multiplying that again by the Value of the Sum expected. And the same Rule is applicable to the happening or failing of as many Events as may be assigned.

Corollary.

If we make abstraction of the Value of the Sum to be obtained, the bare Probability of obtaining it, will be the product of the several Probabilities of happening, which evidently appears from this 8th Art. and from the Corollary to the 5th.

Hitherto, I have confined myself to the consideration of Events independent; but for fear that, in what is to be said afterwards, the terms independent or dependent might occasion some obscurity, it will be necessary, before I proceed any farther, to settle entirely the notion of those terms.

Two Events are independent, when they have no connection one with the other, and that the happening of one neither forwards nor obstructs the happening of the other. Two Events are dependent, when they are so connected together as that the Probability of either's happening is altered by the happening of the other. . . .