3.185 Problem Set 1

Math Review

Due Monday September 9, 2002

1. Calculate the dot product (scalar) and outer product (matrix) for the vectors $(10, 5, 6)$ and $(3, 4, 5)$. (10)

2. For the time-dependent temperature field:

$$T = 400 - 50z \exp\left(-t - x^2 - y^2\right)$$

   (a) Calculate its gradient. (10)

   (b) For the vector field $\vec{u} = 2\hat{\jmath}$, calculate its substantial derivative. (10)

3. "Stagnation" flow in 2-D against a free surface for an incompressible fluid is characterized by the vector flow field

$$u_x = ax, \;\; u_y = -ay$$

   where $\vec{u} = (u_x, u_y)$ is the fluid velocity at a point (x, y) and a is a constant.

   (a) Sketch this vector field for positive y, drawing a few arrows over the range $x \in [-1, 1]$, $y \in [0, 1]$. (5)

   (b) Show that this vector field has zero divergence, so mass is conserved. (7)

   (c) What is the curl of this vector field? (That is, the z-component of the curl.) (8)

4. For the differential equation:

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$$

   (a) Solve for the general solution. There should be three real solutions to the differential equation, though the characteristic polynomial solutions may not be real. (15)

   (b) At $x = 0$, $y = 1$ and $\frac{dy}{dx} = 1$; at $x = \pi$, $y = -1$. What is the linear combination of the solutions to part 4a which satisfies these boundary conditions? (10)

5. For the error function defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi,$$

   calculate:

$$\frac{\partial}{\partial t} \text{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right)$$

   and simplify as much as possible. (15)

6. Show that

$$C = \frac{a}{\sqrt{t}} e^{-\frac{x^2}{4Dt}}$$

   is a solution to the partial differential equation

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}. \text{ (10)}$$