3.185 Test 1 (Amended)

Diffusion, Heat Conduction

October 10-17, 2003

Welcome to the first 3.185 test of Fall 2003. Do not open this until you are told to begin.

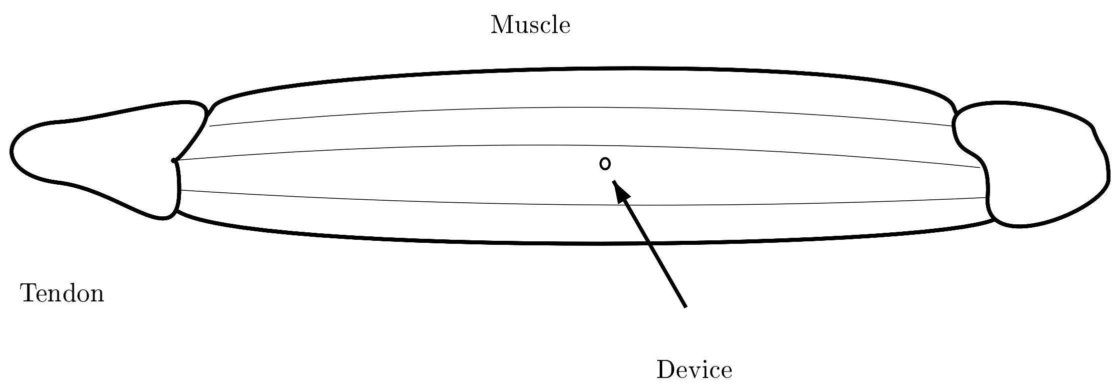
- Total time for answering test questions is 50 minutes.
- This is an in-class, closed-notes, closed book test, and you may not consult others during it, though you may use a calculator.
- You are expected to know and be able to use the equations on the overview sheet. Other equations are given on the last page of this exam. If you need an equation which is neither on the review sheet nor on the equations page, ask and it will be written on the board.
- Feel free to use this test booklet as scratch paper, you can take it with you after the test, and you may ask for extra scratch paper or answer booklets at any time.
- Answer all of the questions, and be sure to show *all* work in your answer booklets, so if your numerical answer is incorrect, you might get partial credit for correct methodology and equations.
- Please begin your answers for each question on a new sheet of paper.
- You may answer the questions in any order.
- The test will be graded and scored, then returned to you along with a fresh test so you can correct it during the Thursday or Friday recitation. You may take as long as you like to correct the test, and if logistical difficulties prevent you from completing it in recitation, we will make arrangements for you to complete it at another time.
- Please indicate clearly in the Section space on the cover of your answer booklet when you will make your corrections, e.g. "Thursday recitation" or "Friday recitation".
- You may use any resources you like to help you to understand the material between the in-lecture portion and correction during recitation, including the instructor and TA (though we might not tell you exactly how to answer a question).
- Following the corrections, your test will be re-graded and scored. Your final score for the test will be the mean of these two scores.

Knock it dead!

1. Write your name on all of your answer booklets (5 pts)

2. Macromolecule diffusion into muscle tissue (35 pts)

A spherical macromolecular drug delivery device 1mm in diameter is implanted into the middle of a large muscle and slowly diffuses its payload into the surrounding tissue. We want to know something about the region of muscle with very high concentration of the drug.



You may assume:

- The device releases the drug at a consistent rate for sufficient time that diffusion in the muscle reaches quasi-steady state.
- The muscle is so much larger than the drug delivery device that it can be considered "infinite".
- (Parts 2a through 2c) diffusion in muscle is isotropic with average diffusivity D.

(a) Show that $C = A/r + B$ is a solution to the steady-state isotropic diffusion equation with spherical symmetry, where A and B are arbitrary constants. (8)

(b) At the surface of the device $r = 0.5$mm, the concentration is known, call it $C = C_0$, and an "infinite" distance away (the outer surface of the muscle, assume it's a long distance away), the concentration is zero. Find the values of A and B which fit these boundary conditions. (8)

(c) What is the size and shape of the region where the concentration is at least $C_0/2$? (6)

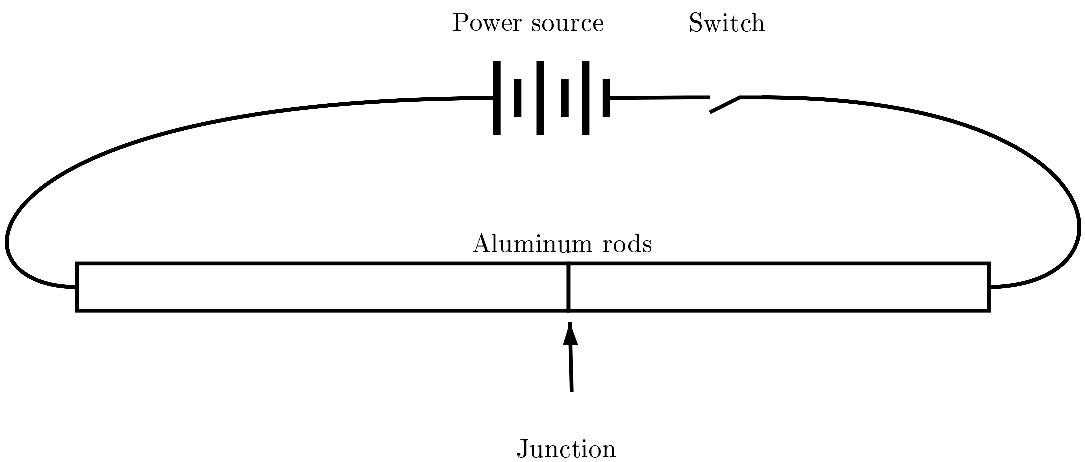
(d) Real muscle is fibrous, with faster diffusion along the length of the cells than across them, thus not isotropic. Will this fact change your answer to part 2c? Why or why not? (6)

(e) Name at least one real-world biological factor which might complicate this diffusion problem, and briefly describe its effect. (Hint: what mechanisms of mass transfer have we discussed in class?) (7)

3. Heat Transfer in Resistance Welding (33 pts)

In resistance welding, two pieces of metal are held against each other, and a very high voltage is applied between them for a very short time. The highest resistance in the circuit is at the interface, so heat concentrates there, melting the metal and joining the two pieces together.

Two cylindrical aluminum rods are joined end-to-end in this way. You would like to know the temperature profile as a function of time $T(x,t)$ to assess the width of the heat-affected zone (region with significant microstructure change due to heating).



Aluminum data:

- Thermal conductivity: $k = 238 \frac{\mathrm{W}}{\mathrm{m \cdot K}}$
- Density: $\rho = 2700 \frac{\mathrm{kg}}{\mathrm{m^3}} (= 2.7 \frac{\mathrm{g}}{\mathrm{cm^3}})$
- Heat capacity: $c_p = 917 \frac{\mathrm{J}}{\mathrm{kg \cdot K}}$

(a) Let $t = 0$ at the moment immediately after the very brief application of current, when it has just been shut off. Sketch the temperature distribution at several times starting at $t = 0$. (8)

(b) Assuming there is no heat loss from the sides of the rod, write the heat conduction equation solution which describes the temperature distribution at moderate to long time scales (while the rod length can still be considered infinite). (7)

(c) At $t = 0$, the whole rod is at 40°C due to Joule heating by the current. Another 3×10^6 J/m^2 of Joule heating energy has been deposited right at the junction to make the weld. Calculate the maximum temperature at $t = 1$ seconds. (Hint in applying this solution: consider the relationship between energy or enthalpy and temperature.) (11)

(d) Also at $t = 1$ seconds, what is the width of the region where the temperature difference $T - 40$°C is at least half of its maximum value? (9)

4. Time scales (25 pts)

   Steel data:

   - Thermal conductivity: $k = 30 \frac{\mathrm{W}}{\mathrm{m \cdot K}}$
   - Density: $\rho = 7600 \frac{\mathrm{kg}}{\mathrm{m^3}}$
   - Heat capacity: $c_p = 700 \frac{\mathrm{J}}{\mathrm{kg \cdot K}}$
   - Approx. heat transfer coefficients: $h = 100 \frac{\mathrm{W}}{\mathrm{m^2 \cdot K}}$ in air, $6000 \frac{\mathrm{W}}{\mathrm{m^2 \cdot K}}$ in water.

   (a) Define the time scales to steady-state of diffusion and heat conduction and give an expression for each. (6)

   (b) For a given material and geometry, is the time scale larger when the Biot number is large or small? Briefly explain. (5)

   (c) A steel plate 0.02m thick leaves a heat treating furnace at 425°C and cools from the top only in air at 25°C (the bottom of the plate rests on an insulating surface). How long does it take for the maximum temperature to reach 65°C? (7)

   (d) Instead of cooling in air, the plate is plunged into water and cooled from both sides. This time, how long does it take for the maximum temperature to reach 65°C? (7)

Equation sheet

- Unsteady-state diffusion equation with spherical symmetry:

$$\frac{\partial C}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D \frac{\partial C}{\partial r}\right) + G.$$

- Shrinking Gaussian solution to the heat equation:

$$T = T_i + \frac{(T_0 - T_i)\delta}{\sqrt{\pi \alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right).$$

- Fourier series solution to the heat equation:

$$T = T_s + (T_i - T_s)\sum_{n=0}^{\infty} a_n \exp\left(-\frac{n^2\pi^2\alpha t}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right),$$

  for a square wave with period $2L$, the a_n coefficients are given by:

$$a_n = \frac{4}{n\pi}.$$

- Newtonian cooling:

$$\frac{T - T_{fl}}{T_i - T_{fl}} = \exp\left(-\frac{Aht}{V\rho c_p}\right).$$

- Plate center cooling curves ($n = x/x_1 = 0$ at the center, $m = k/hL = 1/\mathrm{Bi}$, $X = \frac{\alpha t}{x_1^2}$; note that its log-linear scale makes extrapolation pretty accurate):

Insert $n = 0$ plate graph from W^{3}C Appendix F.